Orthogonality and Characterizations of Inner-Product Spaces

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Orthogonality and Characterizations of Inner-Product Spaces

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Dedicated

to

My Parents

Certificate

This is to certify that the matter embodied in the thesis entitled "Orthogonality and Characterizations of Inner-Product Spaces" by Jagadish Prasad for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried by him under my supervision and guidance. The thesis has, in my opinion, reached the standard fulfilling the requirements of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

O. P. Kapoor

Dated: November 13,1978. *(O.P. Kapoor)*

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Jagadish Prasad

Dated : November 13, 1978. Jagadish Prasad

CONTENTS

SYNOPSIS

The thesis entitled " Orthogonality and Characterizations of Inner-Product Spaces" is devoted to a study of various orthogonalities - Roberts, Isosceles, Pythagorean, Birkhoff - James, Generalized inner-product and N-orthogonality leading to

(i) some new criteria of strict convexity and smoothness of normed linear spaces,

(ii) some new characterizations, and new proofs for old characterizations of inner-product spaces.

Continuous orthogonality vector spaces - a modified form of orthogonality vector spaces of Gudder and Strawther[(1)] are introduced. Finally results on orthogonally additive functionals on lines similar to those in Sundaresan and Kapoor[(2)] have been obtained.

The thesis consists of five chapters.

Chapter I - the introductory chapter, includes various definitions and important results which have been used in later chapters. An attempt has been made to make the thesis as self-contained as possible.

Chapter II of the thesis deals mostly with characterizations of inner-product spaces amongst normed linear spaces using norm identities and the orthogonality notions. Some of the results of

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(1) Pac. J. Math. 58 (1975), 427-436.

(2) Canad. J. Math. 25 (1973), 1121-1132.

this chapter are :

<u>Theorem</u>. For a normed linear space X,

(i) isosceles orthogonality is unique if and only if X is strictly convex,

(ii) Pythagorean orthogonality is unique.

<u>Theorem</u>. In a normed linear space, each of the following conditions is necessary as well as sufficient for the space to be an inner-product space.

(i) Birkhoff-James orthogonality implies isosceles orthogonality i.e.

$||x + \lambda y|| \geq ||x||$ for all $\lambda \in R \implies ||x+y|| = ||x-y||$.

(ii) Isosceles orthogonality implies Birkhoff-James orthogonality i.e.

$||x+y|| = ||x-y|| \implies ||x + \lambda y|| \geq ||x||$ for all $\lambda \in R$.

(iii) Pythagorean orthogonality implies isosceles orthogonality i.e.

$||x+y||^2 = ||x||^2 + ||y||^2 \implies ||x+y|| = ||x-y||$.

(iv) Isosceles orthogonality implies Pythagorean orthogonality i.e.

$||x+y|| = ||x-y|| \implies ||x+y||^2 = ||x||^2 + ||y||^2$.

(v) Birkhoff-James orthogonality implies Pythagorean orthogonality i.e.

$||x + \lambda y|| \geq ||x||$ for all $\lambda \in R \implies ||x+y||^2=||x||^2+||y||^2$.

(vi) Pythagorean orthogonality implies Birkhoff-James orthogonality i.e.

$||x+y||^2 = ||x||^2 + ||y||^2 \implies ||x+\lambda y|| \geq ||x||$ for all $\lambda \in R$.

<u>Theorem</u>. Let X be a normed linear space and $0 < a, b < 1$, The following are equivalent :

(i) X is an inner-product space.

(ii) $x,y \in X$ and $||x+y||^2 + ||ax+by||^2 = ||ax+y||^2 + ||x+by||^2$ $\Longrightarrow ||x+\lambda y|| \geq ||x||$ for all $\lambda \in R$.

(iii) $x,y \in X$ and $||x+\lambda y|| \geq ||x||$ for all $\lambda \in R \Longrightarrow ||x+y||^2 + ||bx+ay||^2 = ||bx+y||^2 + ||x+ay||^2$.

<u>Theorem</u>. Let X be a normed linear space X and $0 < a, b < 1$. Consider the following statements :

(i) X is an inner-product space.

(ii) $x,y \in X$ and $||x+y||^2 + ||ax+by||^2 = ||ax+y||^2 + ||x+by||^2$ $\Longrightarrow ||x+y|| = ||x-y||$.

(iii) $x,y \in X$ and $||x+y|| = ||x-y|| \Longrightarrow ||x+y||^2 + ||ax+by||^2 = ||ax+y||^2 + ||x+by||^2$.

Then (i) ==> (ii) ==> (iii), and (iii) ==> (i) when $a = b$.

<u>Theorem</u>. A normed linear space X is an inner-product space if and only if either for all $x, y, z \in X$

$$||y+z-2x||^2 + ||z+x-2y||^2 + ||x+y-2z||^2 = 3\left[||y-x||^2 + ||x-z||^2 + ||z-y||^2\right]$$

or for all x, y, z and $w \in X$

$$||y-x||^2 + ||z-y||^2 + ||w-z||^2 + ||x-w||^2 = ||z-x||^2 + ||w-y||^2 + 4\left|\left|\frac{x+z}{2} - \frac{y+w}{2}\right|\right|^2.$$

<u>Theorem</u>. A normed linear space is an inner-product space if and only if the following holds :

(L') for some $\gamma \neq 0,1$, $||x+y||^2 = ||x||^2 + ||y||^2 \Longrightarrow ||x+\gamma y||^2$

$$= ||x||^2 + ||\gamma y||^2.$$

<u>Theorem</u>. Let X be a normed linear space and $p_i \neq 0 \neq q_i$, $i = 1,2,\ldots,n$ be numbers such that $p_i\, q_i > \sum_{j\neq i} p_j\, q_j$ for some i. Then X is an inner-product space if and only if the following holds

$$||x+\lambda y|| \geq ||x|| \text{ for every } \lambda \in R \Longrightarrow \sum_{i=1}^{n} p_i ||x+q_i y||^2$$

$$= \sum_{1}^{n} p_i\, ||x-q_i y||^2.$$

Chapter III deals with the generalized inner-product $\langle x,y\rangle$ which is the right Gâteaux derivative of the functional $\frac{1}{2}\,||x||^2$, at x in the direction of y. The orthogonality for the generalized inner-product is $x \perp_G y \Longleftrightarrow \langle x,y\rangle = 0$. In this connection some of the results in the thesis are given below.

<u>Theorem</u>. A normed linear space X is smooth if and only if Birkhoff-James orthogonality implies G-orthogonality and X is strictly convex if and only if the G-orthogonality is left unique.

Alternative and some what simpler proofs have been given for the following results.

<u>Theorem</u>[3]. If in a normed linear space X the G-orthogonality is symmetric, then the Birkhoff-James orthogonality

(3) R.C. James, Trans. Amer. Math. Soc. 61 (1947), 265-292.

is also symmetric and X is both strictly convex and smooth.

Theorem[4]. For a normed linear space X, the following are equivalent :

(i) X is an inner-product space.

(ii) $||x|| = ||y|| \implies \lim_{n \to \infty} (||nx+y|| - ||x+ny||) = 0$.

(iii) $\langle x,y \rangle = \langle y,x \rangle$ for all $x,y \in X$.

(iv) $\langle x,y \rangle$ is linear in x for each $y \in X$.

Theorem[5]. A normed linear space X, with dimension $X \geq 3$, is an inner-product space if $\langle x,y \rangle = 0 \implies \langle y,x \rangle = 0$.

The main result of Chapter IV is :

Theorem. Let X be a continuous orthogonally vector space and f be a nontrivial real valued function on X such that

(i) $f(\lambda x) = |\lambda|\, f(x)$ for $\lambda \in R$.

(ii) f is orthogonally increasing.

Then f is a norm on X.

Further if

(iii) f is also Gâteaux differentiable, then the orthogonality is Birkhoff-James orthogonality.

The last Chapter V deals with the N-orthogonality defined on a Hausdorff locally convex linear topological space X as follows : Let X^* be the dual of X and let $N : X \to X^*$ be a nonlinear mapping. $x \in X$ is said to be N-orthogonal to

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(4) R.A. Tapia, Proc. Amer. Math. Soc. 41 (1973), 569-574.

(5) D. Laugwitz, Proc. Amer. Math. Soc. 50 (1975), 180-188.

$y \in X$ ($x \perp_N y$, in short) whenever the value of Nx at y denoted by (Nx,y) is zero.

The theorem given below gives sufficient conditions on the mapping N so that each two-dimensional subspace of X contains a pair of nonzero N-orthogonal elements.

<u>Theorem.</u> Let $N : X \to X^*$ be a hemi-continuous mapping such that $(Nx,x) > 0$ for $x \neq 0$. Let the N-orthogonality be positive left homogeneous. Then for any two linearly independent elements $x,y \in X$, there exist numbers b and c such that $x \perp_N bx+y$ and $cx+y \perp_N x$.

Among other results in this chapter we have the following representation theorems :

<u>Theorem.</u> Let $N : X \to X^*$ be a hemi-continuous mapping with the property $(Nx,x) > 0$ for $x \neq 0$, and let the N-orthogonality be homogeneous and nonsymmetric. If f is a continuous N-orthogonally additive functional on X, then f is linear.

<u>Theorem.</u> Let $N : X \to X^*$ be a hemi-continuous monotone mapping. Let N-orthogonality be symmetric. Then an odd continuous functional is N-orthogonally additive if and only if it is linear.

And finally

<u>Theorem.</u> If X is a locally convex space and X^* is the dual of X and $N : X \to X^*$ is a mapping satisfying the following properties :

(i) $(Nx,x) > 0$ and $(Nx,x) = 0 \iff x = 0$,

(ii) N is hemi-continuous,

(iii) N-orthogonality satisfies the Pythagorous property i.e.

$$(N(x+y),\ x+y) = (Nx,x) + (Ny,y),$$

(iv) $N(tx) = t\,N(x)$ for $x \in X$ and $t \in R$.

Then X is an inner-product space in the sense that the N-orthogonality is an inner-product orthogonality, and N is a linear transformation.

CHAPTER I

Introduction and Preliminaries

A normed linear space X is called an inner-product space if there is an inner-product defined in it such that

$$||x||^2 = (x,x),$$

where (x,y) stands for the inner-product of x and y. There are many properties known for inner-product spaces which are not true for all normed spaces; many of these are sufficiently strong so as to be characteristics of inner-product spaces. The best known characterization of inner-product spaces among normed linear spaces is by Jordan and von Neumann [24] which states :-

(JN) For every pair x,y of elements of X,

$$||x+y||^2 + ||x-y||^2 = 2 \left[||x||^2 + ||y||^2 \right],$$

that is, in any parallelogram the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the sides.

Day has included in his book [6] other known important characterizing properties of inner-product spaces. Those which interest us are given below for later reference.

(JN_1) A normed linear space X is an inner-product space if and only if every two-dimensional subspace is Euclidean.

(F) If $||x|| = ||y||$, then for all real numbers α and β,

$$||\alpha x + \beta y|| = ||\beta x + \alpha y|| \quad \text{(Ficken [10])}.$$

This can be restated as :

(F') If $||x+y|| = ||x-y||$, then for all real λ,

$$||x + \lambda y|| = ||x - \lambda y|| \quad \text{(James [19])}.$$

(L) There is a fixed number $\gamma \neq 0, \pm 1$ such that $x, y \in X$ and $||x+y|| = ||x-y||$ imply

$$||x + \gamma y|| = ||x - \gamma y|| \quad \text{(Lorch [31])}.$$

(D_1) If $||x|| = ||y|| = 1$, then

$$||x+y||^2 + ||x-y||^2 = 4 \quad \text{(Day [7])}.$$

($S, \sim$) If $||x|| = ||y|| = 1$, then

$||x+y||^2 + ||x-y||^2 \sim 4$, where $\sim$ is one of the relations $=$, $\geq$ or $\leq$ (Schoenberg [38]).

($D, \sim$) If $x, y \in X$ and $||x|| = ||y|| = 1$. then there exist λ and μ with $0 < \lambda < 1$ and $0 < \mu < 1$, such that

$$\mu(1-\mu)\, ||\lambda x + (1-\lambda)y||^2 + \lambda(1-\lambda)||\, \mu x - (1-\mu)y||^2$$

$$\sim [\lambda + \mu - 2\lambda\mu][\lambda\mu + (1-\lambda)(1-\mu)],$$

where $\sim$ is one of the relations $=$, $\geq$ or $\leq$ (Kasahara's theorem improved by Day [8]).

(E) The set of points of norm one in each plane through 0 is an ellipse.

(M) If $||x+y|| = ||x-y||$, then for all real λ,

$$||x + \lambda y|| \geq ||x||.$$

The proof of these characterizations except that of (E) is given in the book. The proof of (E) is not easily available in literature. We will give a proof of (E) at the end of this chapter. There are other known characterizations of inner-product spaces which will be of interest to us. Some of them are best stated in terms of various notion of orthogonality in normed linear spaces. We will defer them and others for a little while and give below basic definitions and notations for convenience of reference.

Throughout the thesis, we will use the reals R as the scalars. The symbols $||x||$ and $q(x)$ will be used alternatively for the norm of an element x. The <u>right</u> (<u>left</u>) <u>Gâteaux derivative</u> of the norm functional q in the direction of y is

$$q'_+(x,y) = \lim_{t \to 0^+} \frac{||x+ty|| - ||x||}{t}.$$

$$(q'_-(x,y) = \lim_{t \to 0^-} \frac{||x+ty|| - ||x||}{t}.)$$

The above two limits exist because q is a convex functional. If

$$q'_+(x,y) = q'_-(x,y),$$

we write the common value as $q'(x,y)$ and call it the Gâteaux derivative of the norm at x in the direction of y.

If f is a convex functional on a locally convex space X with continuous dual X^*, then a subgradient of f at x is a $\phi \in X^*$ such that

$$f(y) \geq f(x) + \phi(y-x), \text{ for all } y \in X,$$

and the set

$$\partial f(x) = \{\phi \in X^* : \phi \text{ is a subgradient of } f \text{ at } x\}$$

is called the subdifferential of f at x. One can easily see for a normed linear space X, the subdifferential of the norm is

(1.1.1) $$\partial q(x) = \{\phi \in X^* : ||\phi|| = 1 \text{ and } \phi(x) = ||x||\}$$

and that $\partial q(x)$ is a nonempty weak * compact convex subset of X^*. It is known that

$$q'_+(x,y) = \max\{\phi(y) : \phi \in \partial q(x)\}$$

(1.1.2) and

$$q'_-(x,y) = \min\{\phi(y) : \phi \in \partial q(x)\} .$$

This is immediate from the following (see [17], page 27).

Theorem 1.1.1. (Moreau, Pshenichnii) Let X be a real locally convex space and f be a convex functional. Assume f is continuous at x_0. Then for all $x \in X$,

$$f'_+(x_0,x) = \max\{\phi(x) : \phi \in \partial f(x_0)\}$$

and

$$f'_-(x_0,x) = \min\{\phi(x) : \phi \in \partial f(x_0)\}.$$

The following theorem describes wellknown properties of the one sided derivatives of the norm functional.

Theorem 1.1.2. Let $0 \neq x$, y and $z \in X$, a normed linear space and let $\lambda \in R$ and $\mu \geq 0$. Then

(a) $q'_+(x,y+z) \leq q'_+(x,y) + q'_+(x,z)$

(b) $q'_+(x,\mu y) = \mu q'_+(x,y)$

(b') $q'_+(\lambda x,y) = q'_+(x,y), \ \lambda > 0$

$\qquad = -q'_-(x,y), \ \lambda < 0$

(c) $q'_+(x,y) = -q'_-(x,-y)$

(d) $q'_-(x,y) \leq q'_+(x,y)$

(e) $q'_+(x,-)$ is a linear functional if and only if the norm is Gâteaux differentiable at x.

(f) $|q'_+(x,y)| \leq ||y||$

(g) $q'_+(x,\lambda x + \mu y) = \lambda||x|| + \mu q'_+(x,y)$.

Proof. The properties (a) to (e) hold for right Gâteaux derivative of any convex functional and are proved in various text books e.g. Köthe [27]. Property (f) can be easily seen from the equations (1.1.1) and (1.1.2) above. For (g), we have from (1.1.1) and (1.1.2)

$$\begin{aligned} q'_+(x, \lambda x + \mu y) &= \max \ \{\phi(\lambda x + \mu y) : \phi \in \partial q(x)\} \\ &= \max \{\lambda\phi(x) + \phi(\mu y) : ||\phi|| = 1, \ \phi(x) = ||x||\} \\ &= \max \{\lambda||x|| + \phi(\mu y) : ||\phi|| = 1, \ \phi(x) = ||x||\} \\ &= \lambda||x|| + \mu \max \{\phi(y) : \phi \in \partial q(x)\} \\ &= \lambda||x|| + \mu \ q'_+(x,y). \end{aligned}$$

A normed linear space X is called smooth if at each point $x \neq 0$, the norm is Gâteaux differentiable. Equivalently there is only one supporting hyperplane of the unit ball $U = \{x : ||x|| \leq 1\}$ at each point of its boundary $S = \{x : ||x|| = 1\}$. X is called strictly convex if each point of S is an extreme point of U. An extreme point of U is a point of U that is not an interior point of any line segment in U. It is easily seen that any line segment in S is contained in a line segment in S with ends as extreme points of U.

The natural definition of orthogonality relation between elements of an inner-product space is that $x \perp y$ if and only if the inner-product (x,y) is zero. In normed linear spaces other notions of orthogonality have been given which are equivalent to this in case the norm arises from an inner-product. We will describe some of them in some detail here.

(i) Roberts' orthogonality ([37, p. 56]) : $x \perp_R y$ (x is orthogonal to y in the sense of Roberts) if

$$||x + \lambda y|| = ||x-\lambda y|| \text{ for all } \lambda \in R.$$

This definition has the weakness that for some normed linear spaces at least one of every pair of orthogonal elements would have to be zero [19, example 2.1].

(ii) Birkhoff-James Orthogonality : x is orthogonal to y in the sense of Birkhoff and James ($x \perp_J y$) if and only if

$$||x + \lambda y|| \geq ||x|| \quad \text{for all } \lambda \in R.$$

This definition was used by Birkhoff [1] and by Fortet [11, 12]. Later on James [20] studied this orthogonality relating it to other concepts such as strict convexity, smoothness, weak compactness, linear functionals and hyperplanes. Some writers have called this orthogonality 'James Orthogonality'.

(iii) Isosceles Orthogonality : $x \perp_I y$ if and only if

$$||x+y|| = ||x-y||.$$

(iv) Pythagorean Orthogonality : $x \perp_P y$ if and only if

$$||x+y||^2 = ||x||^2 + ||y||^2.$$

The above two notions of orthogonality were also suggested by James [19]. One important feature shared by the Birkhoff-James, isosceles and Pythagorean orthogonalities is that every two-dimensional subspace containing a point x, contains a point x' such that x is orthogonal to x'. In other words we have :

<u>Theorem 1.1.3.</u> Let X be a normed linear space and $x \neq 0$ and $y \in X$. There exist numbers α_1, α_2 and α_3 such that

$$x \perp_J \alpha_1 x+y, \quad x \perp_I \alpha_2 x+y \text{ and } x \perp_P \alpha_3 x+y .$$

This result does not hold in the case of Roberts' Orthogonality. In fact we have :

<u>Theorem 1.1.4.</u> For any $x \neq 0$ and $y \in X$, there exists a number α such that $x \perp_R \alpha x+y$ if and only if X is an inner-product space.

<u>Remark 1.1.5.</u> This theorem has been stated without proof by James as a corollary to theorem 4.7 in [19, p. 298]. We will give a proof of this theorem towards the end of this chapter for completeness sake. In fact we will be using it in one of our theorems in the next chapter.

An orthogonality $\perp$ is called <u>right</u> (<u>left</u>) unique if for each $x \neq 0$ and $y \in X$, there exists only one α such that $x \perp \alpha x+y$ ($\alpha x+y \perp x$). It is called <u>unique</u> if it is left as well as right unique.

The following results about the orthogonalities will be needed later.

<u>Theorem 1.1.6.</u> (James [20]) In a normed linear space Birkhoff-James Orthogonality is right (left) unique if and only if the space is smooth (strictly convex).

<u>Theorem 1.1.7.</u> (James [19]) If $x \neq 0$ and y are elements of a normed linear space X and $||y|| \leq ||x||$, then $x \perp_I \alpha x+y$ implies that

$$|\alpha| \leq \frac{||y||}{||x||} .$$

<u>Theorem 1.1.8.</u> Let $x \neq 0$ and $y \in X$. Then $x \perp_J y$ if and only if

$$q'_+(x,y) \geq 0$$

$$\text{and} \quad q'_-(x,y) \leq 0.$$

Proof. Suppose $x \perp_J y$. We have

$$||x + \lambda y|| \geq ||x|| \quad \text{for all } \lambda \in R.$$

Therefore

$$\frac{||x + \lambda y|| - ||x||}{\lambda} \geq 0 \quad \text{for } \lambda > 0$$

$$\text{and} \quad \frac{||x + \lambda y|| - ||x||}{\lambda} \leq 0 \quad \text{for } \lambda < 0.$$

Thus

$$q'_+(x,y) = \lim_{\lambda \to 0^+} \frac{||x + \lambda y|| - ||x||}{\lambda} \geq 0$$

$$\text{and} \quad q'_-(x,y) = \lim_{\lambda \to 0^-} \frac{||x + \lambda y|| - ||x||}{\lambda} \leq 0.$$

The other way also follows from the fact that

$$\frac{||x + \lambda y|| - ||x||}{\lambda}$$

is monotonically increasing for all positive and negative λ's.

An orthogonality $\perp$ is called <u>right</u> (<u>left</u>) <u>homogeneous</u> if

$$x \perp y \Longrightarrow x \perp \lambda y \text{ for all } \lambda \in R$$

$$(x \perp y \Longrightarrow \lambda x \perp y \text{ for all } \lambda \in R).$$

It is homogeneous if it is both right and left homogeneous. It is called symmetric if $x \perp y \Longrightarrow y \perp x$ and right (left) additive if

$$x \perp y \text{ and } x \perp z \Longrightarrow x \perp y+z$$

$$(y \perp x \text{ and } z \perp x \Longrightarrow y+z \perp x).$$

The results given below are useful to us. Some of these give further necessary and sufficient conditions on a normed linear space to be an inner-product space.

Theorem 1.1.9. (Birkhoff [1], James [21], Day [7])

Let X be a normed linear space of dimension greater than two. Then it is an inner-product space if and only if Birkhoff-James orthogonality is symmetric.

Theorem 1.1.10. A normed linear space is an inner-product space if and only if the isosceles (Pythagorean) orthogonality is either homogeneous or additive (James [19]).

Theorem 1.1.11. In a normed linear space X, the following are equivalent :

(a) X is an inner-product space.

(b) $x \perp_I y \Longrightarrow x \perp_P y$.

(c) $x \perp_P y \Longrightarrow x \perp_I y$.

(d) $x \perp_I y \Longrightarrow x \perp_J y$.

Remark 1.1.12. Day [7] proved that (a) <==> (b), (a) <==> (c). (d) is nothing but the criterion (M) on page 155 [6]. For proving the equivalence of (a) and (c), Day [7] first proves that the space is uniformly convex with modulus of convexity $\delta(\epsilon) = 1 - (1 - \frac{\epsilon^2}{4})^{1/2}$. We will, however, give a different proof of this theorem in the second chapter of this thesis.

(v) Carlsson Orthogonality : Carlsson [4] gave a definition of orthogonality which includes the isosceles and Pythagorean orthogonalities as special cases.

An element x of a normed linear space X is said to be orthogonal to an element $y \in X$ in the sense of Carlsson (in symbol $x \perp_C y$) if

$$\sum_{r=1}^{m} a_r \,||b_r\, x + c_r\, y||^2 = 0,$$

where a_r, b_r and c_r, $r = 1,2,\ldots,m$ are fixed real numbers satisfying the relations

$$\sum_{r=1}^{m} a_r\, b_r^2 = 0 = \sum_{r=1}^{m} a_r\, c_r^2 \text{ and } \sum_{r=1}^{m} a_r\, b_r\, c_r = 1.$$

Carlsson's orthogonality is said to satisfy the condition (H) if

$$x \perp_C y \Longrightarrow \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{m} ||b_r\, nx + c_r\, y||^2 = 0.$$

The following characterizations of inner-product spaces are due to Carlsson [4,5].

Theorem 1.1.13. Let X be a normed linear space with a Carlsson's orthogonality $\perp_C$. Then X is an inner-product space if and only if $\perp_C$ satisfies the condition (H).

Theorem 1.1.14. Let $a_\nu \neq 0$, b_ν, c_ν, $\nu = 1,2,\ldots,m$ be real numbers such that (b_ν, c_ν) and (b_μ, c_μ) are linearly independent for $\nu \neq \mu$. If X is a normed linear space satisfying condition

$$\sum_{\nu=1}^{m} a_\nu \, ||b_\nu u + c_\nu v||^2 = 0 \text{ for all } u \text{ and } v \in X,$$

then X is an inner-product space.

Theorem 1.1.15. Let $a_\nu \neq 0$, b_ν, c_ν, $\nu = 1,2,\ldots,m$ be a fixed collection of real numbers satisfying

$$\sum_{\nu=1}^{m} a_\nu b_\nu^2 = \sum_{\nu=1}^{m} a_\nu c_\nu^2 = \sum_{\nu=1}^{m} a_\nu b_\nu c_\nu = 0$$

and such that

$$(b_\nu, c_\nu) \text{ and } (b_\mu, c_\mu)$$

are linearly independent for $\mu \neq \nu$.

If X is a normed linear space satisfying the condition

$$\sum_{\nu=1}^{m} a_\nu \, ||b_\nu x + c_\nu y||^2 \geq 0 \quad \text{for all } x,y \in X,$$

then X is an inner-product space.

In the second chapter of this thesis we will study the inter-relationship between various orthogonalities, obtain some

new criteria for normed spaces to be inner-product spaces. We provide new proofs for known characterizations also.

Let us, for instance, take the simple proposition in plane geometry :

Suppose ΔABC is a right-angled triangle, right-angled at A and points P and Q are any points on AB and AC respectively. Then

$$BQ^2 + PC^2 = BC^2 + PQ^2.$$

The analogue of this in an inner product space is :

$$(x,y) = 0 \iff ||ax+y||^2+||x+by||^2=||ax+by||^2+||x+y||^2$$

$$\text{for } 0 < a,\ b < 1.$$

A question that we look into in the second chapter is whether in the setting of a normed linear space, the presence of such a proposition using one of the notions of orthogonality mentioned earlier forces the space to be an inner-product space or not. By way of another example we will prove the following criteria as analogues of the criteria (L) and (M) (see pages 2 and 3).

(L') There is a fixed number $\gamma \neq 0, \pm 1$, such that $x,y \in X$ and

$$||x+y||^2 = ||x||^2 + ||y||^2 \text{ imply } ||x+\gamma y||^2 = ||x||^2 + ||\gamma y||^2.$$

(M') If $x, y \in X$, then

$$||x + \lambda y|| \geq ||x|| \quad \text{for all } \lambda \in R \text{ implies}$$
$$||x + y|| = ||x-y||.$$

Lorch [31] gives a number of characterizations based on simple properties of Euclidean geometry and observes that almost any one of the strictly metrical classical theorems will characterize inner-product spaces. Using some more elementary propositions in Euclidean geometry as postulates we obtain new norm postulates as necessary and sufficient conditions for a normed linear space to be an inner-product space. We will also show that some of these norm identities and those given by Johnson [22] and Rakestraw [35] follow from the theorem 1.1.14 of Carlsson mentioned above.

Several characterizations of inner-product spaces among normed linear spaces are known which use the notions of semi-inner product of Lumer [32] and generalized inner-product of Tapia [45]. There are other characterizations which involve differentiability properties of the norm.

A <u>semi-inner-product</u> on a vector space X is a real function $[x,y]$ on $X \times X$ with following properties :

(s_1) $[z, x+y] = [z,x] + [z,y]$, $[x, ty] = t[x,y]$ for all $t \in R$.

(s_2) $[x,x] > 0$ when $x \neq 0$.

(s_3) $\quad |[x,y]|^2 \leq [x,x]\,[y,y]$.

A semi-inner-product space is a normed linear space with $||x|| = [x,x]^{1/2}$. G. Lummer showed that every normed linear space X has a semi-inner product defined by

$$[x,y] = f_x(y) \text{ where } f_x \in X^* \text{ is chosen such that}$$
$$f_x(x) = ||x||^2.$$

Giles proved the following theorem relating the Birkhoff-James orthogonality with a semi-inner-product orthogonality.

Theorem 1.1.16. Let $[x,y]$ be a semi-inner-product on a vector space X, with the following properties :

(a) $[tx,y] = t[x,y]$ for all $t \in R$.

(b) $[x+ty,y] \to [x,y]$ as $t \to 0$.

Then $[x,y] = 0$ if and only if x is Birkhoff-James orthogonal to y.

Let X be a normed linear space. The generalized inner-product $\langle x,y\rangle$ of x with y is defined to be the right Gâteaux derivative of the convex functional $f(x) = \frac{1}{2}||x||^2$ at x in the direction of y. Thus

$$\langle x,y\rangle = f'_+(x,y) = ||x||\, q'_+(x,y).$$

Tapia [45] shows that although this generalized inner-product does not have as much structure as a semi-inner-product, similar to a semi-inner-product it must be inner product whenever it is

linear in x or is symmetric. He deduces that $\langle x,y\rangle$ is an inner-product if and only if the mapping

$$f(x) = \frac{1}{2} ||x||^2$$

is twice Frechet differentiable at the origin. This is of interest in relation to inner-product characterizations by twice Fréchet differentiability of the norm given by Bonic and Reis [3], Rao [36], Sundaresan [42] and Leonard and Sundaresan [30].

In the third chapter we discuss the orthogonality in a normed linear space defined by

$$x \perp_G y \iff \langle x,y\rangle = 0,$$

obtain a result on the right existence of G-orthogonal pairs in every two-dimensional subspace and show by a counter example that the left existence may not be there. We then observe that Laugwitz's characterization [29] of inner product spaces in terms of symmetry of G-orthogonality is, in a way, already proved by James in [20]. In the same chapter we provide an elementary proof of the characterization in terms of linearity or symmetry of generalized inner-product by Tapia mentioned above.

In a paper Gudder and Strawther [15] call a real vector space X, an <u>orthogonality vector space</u> if there is a relation $x \perp y$ on X that satisfies the following postulates :

(O_1) $x \perp 0$ and $0 \perp x$ for all $x \in X$.

(O_2) If x and y are non-zero elements of X and $x \perp y$, then x and y are linearly independent.

(O_3) If $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in R$.

(O_4) If P is a two-dimensional subspace of X, then for every $x \in P$, there exists a non-zero $y \in P$ such that $x \perp y$.

(O_5) If P is a two-dimensional subspace of X, then there exist non-zero u and $v \in P$ such that $u \perp v$ and $u+v \perp u-v$.

It is easily seen that a normed linear space with Birkhoff-James orthogonality is an example of an orthogonality vector space.

A function $f : X \to R$ is called <u>orthogonally additive</u> (<u>orthogonally increasing</u>) if

$$f(x+y) = f(x) + f(y) \quad \text{whenever } x \perp y$$

$$(f(x+y) \geq f(x) \quad \text{whenever } x \perp y).$$

<u>Theorem 1.1.17</u>. (Gudder and Strawther) Let $(X, \perp)$ be an orthogonality vector space. Suppose there exists a nontrivial orthogonally additive functional

$$f : X \to R$$

such that (i) $f(x) = f(-x)$ for every $x \in X$

and (ii) $f(\alpha_n x) \to f(\alpha x)$ whenever $\alpha_n \to \alpha$.

Then the orthogonality is an inner-product orthogonality.

Theorem 1.1.17 is an abstract version of the following

<u>Theorem 1.1.18</u>. Let X be a normed linear space and $f : X \to R$ be an orthogonally additive functional which is even and hemi-continuous. Then

(a) $f \equiv 0$, if X is not an inner-product space,

(b) there exists a $\lambda \in R$ such that

$$f(x) = \lambda||x||^2 \text{ for all } x \in X,$$

if X is an inner-product space (Sundaresan, [43]).

In the fourth chapter we introduce the concept of a continuous orthogonality vector space which turns out to be a special case of the concept of orthogonality vector spaces of Gudder and Strawther. We then obtain a kind of converse to their result characterizing orthogonally increasing functionals on a normed linear space.

Sundaresan and Kapoor [44] defined orthogonality in a topological vector space X as follows :

Let T be a continuous linear transformation from X into X^*. An element $x \in X$ is said to be <u>T-orthogonal</u> to an element $y \in X$ denoted by $x \perp_T y$ if the value of Tx at y denoted by (Tx,y) is zero. Representation of T-orthogonally additive functionals have been the subject of study in Sundaresan and Kapoor [44]. If X is a Banach space of real valued measurable functions on a measure space and if f and $g \in X$, then f is said to be <u>L-orthogonal to</u> g in the lattice theoretic sense ($f \perp_L g$) if the set

$$\{s : f(s)\, g(s) \neq 0\}$$

is of measure zero. Integral representations of L-orthogonally additive functionals has been subject of extensive study. For these and related results we refer to Drewnski and Orlicz [9], Sundaresan [42] and Mizel and Sundaresan [33] and the bibliography cited therein.

In the fifth chapter of the thesis we consider the representation question on lines similar to those in Sundaresan and Kapoor [44]. In the place of linear transformation T we will use a nonlinear transformation N on a locally convex topological vector space X into X^*, having certain monotonicity and continuity properties. N-orthogonality is studied and the question of description of N-orthogonally additive functionals is looked into.

We end this introductory chapter with the proofs of the criterion (E) and of theorem 1.1.4 as promised earlier.

Proof of (E). In view of (JN_1) we can assume that X is two-dimensional. First let X be an inner product space and let x,y be two linearly independent elements in it. The set

$$S = \{(\alpha,\beta) \in R^2 : ||\alpha x + \beta y||^2 = 1\}$$

$$= \{(\alpha,\beta) \in R^2 : \alpha^2||x||^2+\beta^2||y||^2+2\alpha\beta(x,y) = 1\}$$

is an ellipse in view of Cauchy-Schwartz inequality

$$|(x,y)|^2 \leq ||x||^2\, ||y||^2.$$

To prove the converse let us assume without loss of generality that X is the x-y plane and the unit sphere is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1.1.3}$$

Let $z = re^{i\phi}$ be any non-zero point of X and let the ray from origin to z intersect the ellipse in $z' = r'e^{i\phi}$. Then r' is determined by

$$r'^2 \left[\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right] = 1.$$

So that

$$r' = \frac{ab}{\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}}. \tag{1.1.4}$$

Since norm of X is the Minkowski functional of the ellipse, therefore

$$||z|| = r/r' = \frac{r}{ab}\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}. \tag{1.1.5}$$

Now let $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$ be any two points of the plane. Then

$$z_1 + z_2 = Re^{i\psi}$$

where

$$\left[\begin{array}{l} R\cos\psi = r_1\cos\phi_1 + r_2\cos\phi_2 \\ \\ R\sin\psi = r_1\sin\phi_1 + r_2\sin\phi_2 \end{array}\right. \tag{1.1.6}$$

From (1.1.5) and (1.1.6) we have

$$||z_1+z_2||^2 = \frac{b^2(r_1 \cos \phi_1+r_2 \cos \phi_2)^2+a^2(r_1 \sin \phi_1+r_2 \sin \phi_2)^2}{a^2b^2}.$$

Similarly

$$||z_1-z_2||^2 = \frac{b^2(r_1 \cos \phi_1-r_2 \cos \phi_2)^2+a^2(r_1 \sin \phi_1-r_2 \sin \phi_2)^2}{a^2b^2}$$

and thus we have

$$\begin{aligned} ||z_1+z_2||^2 + ||z_1-z_2||^2 &= 2\,\frac{r_1^2 \cos^2 \phi_1+r_2^2 \cos^2 \phi_2}{a^2} + \\ &\quad + 2\,\frac{r_1^2 \sin^2 \phi_1+r_2^2 \sin^2 \phi_2}{b^2} \\ &= 2\Big[r_1^2\,\frac{(b^2 \cos^2 \phi_1+a^2 \sin^2 \phi_1)}{a^2b^2} + \\ &\quad + r_2^2\,\frac{(b^2 \cos^2 \phi_2+a^2 \sin^2 \phi_2)}{a^2b^2}\Big] \\ &= 2\,\big[||z_1||^2 + ||z_2||^2\big]. \end{aligned}$$

Thus $||\cdot||$ of X satisfies (JN) and hence X is an inner-product space.

Proof of theorem 1.1.4. We will first show that X is a strictly convex space. Let $x \neq y \in X$ such that

$$||x|| = ||y|| = ||\tfrac{x+y}{2}|| = 1.$$

By hypothesis there exists a number α such that

$$(1.1.7)\quad ||k\,\frac{3x+y}{4} + \alpha\,\frac{3x+y}{4} + x|| = ||k\,\frac{3x+y}{4} - \alpha\,\frac{3x+y}{4} - x||$$

for all $k \in R$.

Let $k = \alpha$ in (1.1.7). Thus we get

$$||2\alpha\,\frac{3x+y}{4} + x|| = ||x|| = 1 \geq |2\alpha| - 1,$$

so that $|\alpha| \leq 1$.

Again letting $k = \frac{4}{3} + \alpha$ in (1.1.7) we have

$$(1.1.8)\quad ||(2 + \tfrac{3}{2}\alpha)\,x + (\tfrac{1}{3} + \tfrac{\alpha}{2})y|| = ||\tfrac{y}{3}|| = \tfrac{1}{3}\,.$$

If $\alpha \geq -2/3$, (1.1.8) yields

$$\frac{1}{3} = (2\alpha + \frac{7}{3})\,\left|\left|\frac{(2+\frac{3}{2}\alpha)x + (\frac{1}{3} + \frac{\alpha}{2})y}{2\alpha + \frac{7}{3}}\right|\right| = 2\alpha + \frac{7}{3}$$

which gives $\alpha = -1$. But that is not possible for we are assuming $\alpha \geq -2/3$. Thus

$$-1 \leq \alpha \leq -2/3.$$

But then (1.1.8) gives

$$\frac{1}{3} \geq |2 + \frac{3}{2}\alpha| - |\frac{1}{3} + \frac{\alpha}{2}|$$

$$= 2 + \frac{3}{2}\alpha + \frac{1}{3} + \frac{\alpha}{2} = \frac{7}{3} + 2\alpha.$$

That means $\alpha \leq -1$.

Thus $\alpha = -1$ is the only possible value of α in (1.1.7). With $\alpha = -1$, (1.1.7) becomes

$$||k\,\frac{3x+y}{4} + \frac{x-y}{4}|| = ||k\,\frac{3x+y}{4} - \frac{x-y}{4}|| \quad \text{for all } k \in R.$$

We write this as

$$(1.1.9) \qquad ||\frac{3x+y}{4} + k(x-y)|| = ||\frac{3x+y}{4} - k(x-y)|| \quad \text{for all } k.$$

Similarly we can prove

$$(1.1.10) \qquad ||\frac{x+3y}{4} + \beta(x-y)|| = ||\frac{x+3y}{4} - \beta(x-y)|| \text{for all } \beta \in R.$$

Putting $k = \beta - \frac{1}{2}$ in (1.1.9) and then using (1.1.10) we get

$$||(\beta-1)(x-y) - \frac{x+3y}{4}|| - ||\beta(x-y) - \frac{x+3y}{4}|| = 0 \text{ for all } \beta.$$

Or for all $\beta > 1$

$$||(\beta-1)(x-y) - \frac{x+3y}{4}|| - (\beta-1)\,||x-y|| - \Big[\,||\beta(x-y) - \frac{x+3y}{4}|| - \beta\,||x-y||\,\Big] = ||x-y||$$

or

$$\left[\frac{||(x-y) - \frac{x+3y}{4(\beta-1)}|| - ||x-y||}{\frac{1}{\beta-1}}\right] - \left[\frac{||(x-y) - \frac{x+3y}{4\beta}|| - ||x-y||}{1/\beta}\right] = ||x-y|| \text{ for all } \beta > 1.$$

Taking the limit as β tends to infinity, we get

$$||x-y|| = q'_+\,(x-y, -\frac{x+3y}{4}) - q'_+(x-y, -\frac{x+3y}{4}) = 0,$$

which gives the contradiction. Hence X is strictly convex.

Suppose now that there exists a pair of vectors x and y and number $\lambda \neq 0, \pm 1$ such that

$$||x + y|| = ||x - y||$$

but
$$||x + \lambda y|| \neq ||x - \lambda y||.$$

By hypothesis there exists a number $\alpha \neq 0$ such that

$$||x + k(\alpha x + y)|| = ||x - k(\alpha x+y)|| \text{ for all } k \in R.$$

If $\alpha > 0$, take $k = \frac{1}{1+\alpha}$ and get

$$||x + \frac{\alpha}{1+\alpha} x + \frac{1}{1+\alpha} y|| = ||x - (\frac{\alpha}{1+\alpha} x + \frac{1}{1+\alpha} y)||.$$

Since $\frac{\alpha x}{1+\alpha} + \frac{1}{1+\alpha} y$ is a convex combination of x and y

$$||x-y|| = ||x - (\frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha})|| + ||\frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha} - y||.$$

Therefore

$$||x+y|| = ||x + \frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha}|| + ||\frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha} - y||$$

or

$$||\frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha} + x + y-(\frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha})|| = ||\frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha} x + x||$$

$$+ ||y-(\frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha})||.$$

But X is strictly convex, therefore $x + \frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha}$ is a multiple of $y - (\frac{\alpha x}{1+\alpha} + \frac{y}{1+\alpha})$, which gives $\alpha =-1$, a contradiction.

Similarly if $\alpha < 0$, we take $k = \frac{1}{1-\alpha}$. Now $k(\alpha x + y)$ becomes a convex combination of $-x$ and y which yields a contradiction again in the same manner. That proves that the isosceles orthogonality is homogeneous and therefore the space is an inner-product space by Theorem 1.1.10.

The converse of the theorem is obvious.

CHAPTER II

Characterization of Inner-Product Spaces

In this chapter we will add a few more results in the long list of already known results characterizing inner-product spaces among normed linear spaces through norm identities. The best known such characterization is (JN) by Jordan and von Neumann [24]. Considerable work has been done proving that other norm identities also characterize inner-product spaces. Day [7,8], Schoenberg [38], Kasahara [25], Senechalle [39,40,41], John Oman [34] and others have shown that for some identities these results can be improved in three ways :

1. The identity need not hold for all vectors in the space,
2. The identity can be weakened to an inequality,
3. The identity may vary with the choice of vectors as long as the form of the identity is fixed.

Most of the characterizations obtained in this chapter fall in the first category. For example we will show that if a norm identity holds for mutually orthogonal vectors - orthogonal in some sense, then the space must be an inner-product space.

2.1 Orthogonality Notions and Characterization of Inner-Product Spaces.

Several writers [Day, Blumenthal, Holub] have shown

that if in a normed linear space isosceles orthogonality implies Pythagorean Orthogonality, or the other way round, then the space must be an inner-product space. In this section we will provide new results of the same kind, in addition to proving these results in our own way. But before we do that we establish a lemma that gives a new criterion of strict convexity in terms of isosceles orthogonality similar to the one in theorem 1.1.5 given by James [20] in terms of Birkhoff-James orthogonality.

<u>Lemma 2.1.1</u>. A normed linear space is strictly convex if and only if isosceles orthogonality is unique.

Proof. Suppose X is strictly convex but $\perp_I$ is not unique. There exist $x \neq 0$, $z \in X$ and numbers α and β ($\beta \neq \alpha$) such that $x \perp_I \alpha x+z$ and $x \perp_I \beta x+z$. Suppose $\beta > \alpha$. Then we see that $x \perp_I y$ and $x \perp_I \gamma x+y$ where $\gamma = \beta-\alpha$ and $y = \alpha x+z$.

Consider the function ψ given by

$$\psi(\eta) = ||y + \eta x||.$$

ψ is a strictly convex function such that

$$\psi(1) = ||y+x|| = ||y-x|| = \psi(-1) \quad \text{and}$$

$$\psi(\gamma+1) = ||x+(\gamma x+y)|| = ||-x+(\gamma x+y)|| = \psi(\gamma-1)$$

In case $0 < \gamma \leq 2$, we see that

$$\psi(\gamma-1) = \psi\left[\frac{2-\gamma}{2}(-1) + \frac{\gamma}{2}(1)\right] < \psi(1)$$

$$= \psi\left[\frac{\gamma}{2}(\gamma-1) + (1-\frac{\gamma}{2})(\gamma+1)\right] < \psi(\gamma+1)$$

and that is a contradiction.

In case $\gamma > 2$, $\psi(\eta)$ will have two distinct local minima one each in the intervals $[-1, 1]$ and $[\gamma-1, \gamma+1]$. But the function ψ is strictly convex and it can have at the most one point of minimum - a global minimum. Thus again a contradiction is seen. Hence $\gamma = 0$ which implies that $\alpha = \beta$, a contradiction which proves that in a strictly convex space, the iso-orthogonality is unique.

To prove the other side we start with the assumption that the space is not strictly convex. Then we have, for some pair of vectors $x \neq y$ that

$$||x|| = ||y|| = ||\frac{x+y}{2}|| = 1.$$

But then

$$||x+y|| = ||(x+y)+(x-y)|| = ||(x+y)-(x-y)||$$

Or $\qquad ||x'|| = ||x'+y'|| = ||x'-y'||$

where $x' = x+y$ and $y' = x-y \neq 0$.

Then we have

$$||x' + \frac{y'}{2} - \frac{y'}{2}|| = ||x' + \frac{y'}{2} + \frac{y'}{2}|| = ||x' - \frac{y'}{2} - \frac{y'}{2}||.$$

Thus $\frac{y'}{2} \perp_I x' + \frac{y'}{2}$ and $\frac{y'}{2} \perp_I x' - \frac{y'}{2}$.

Hence isosceles orthogonality is not unique. Therefore if isosceles orthogonality is unique, then the space must be strictly convex.

In contrast to the above lemma and the theorem 1.1.5, we have the following result for Pythagorean Orthogonality.

Theorem 2.1.2. In a normed linear space X, Pythagorean orthogonality is unique.

Proof. Let us assume that Pythagorean orthogonality is not unique. Then as in the previous lemma we can assert that there are vectors $x \neq 0$, $y \in X$ and a number $\alpha > 0$ such that $x \perp_P y$ and $x \perp_P \alpha x+y$. Then we have

$$||x+y||^2 = ||x||^2 + ||y||^2$$

and
$$||x+\alpha x+y||^2 = ||\alpha x+y||^2 + ||x||^2.$$

Setting

$$\phi(\eta) = ||y + \eta x||^2$$

we then have

$$\phi(1) = ||x||^2 + \phi(0) \tag{2.1.1}$$

$$\phi(\alpha+1) = \phi(\alpha) + ||x||^2. \tag{2.1.2}$$

Before proceeding further let us first prove that for $0 < \mu < 1$ and $\phi(\eta_1) \neq \phi(\eta_2)$,

$$\phi[\mu\eta_1 + (1-\mu)\,\eta_2] < \mu\phi(\eta_1) + (1-\mu)\,\phi(\eta_2). \tag{2.1.3}$$

Here

$$
\begin{aligned}
\phi[\mu\eta_1 + (1-\mu)\ \eta_2] &= ||y + [\mu\eta_1 + (1-\mu)\ \eta_2]\ x||^2 \\
&= ||\mu\ (y+\eta_1 x) + (1-\mu)\ (y+\eta_2 x)||^2 \\
&\leq \mu^2\ ||y+\eta_1 x||^2 + (1-\mu)^2\ ||y+\eta_2 x||^2 \\
&\qquad + 2\mu(1-\mu)\ ||y+\eta_1 x||\ ||y+\eta_2 x|| \\
&= \mu||y+\eta_1 x||^2 + (1-\mu)||\ y + \eta_2 x||^2 \\
&\quad + (\mu^2-\mu)\ [||y+\eta_1 x||^2 + ||y+\eta_2 x||^2 \\
&\quad - 2||y+\eta_1 x||\ ||y+\eta_2 x||] \\
&= \mu\phi(\eta_1)+(1-\mu)\ \phi(\eta_2) \\
&\quad - \mu(1-\mu)\ [||y+\eta_1 x||-||y+\eta_2 x||]^2 \\
&\leq \mu\phi(\eta_1) + (1-\mu)\ \phi(\eta_2)
\end{aligned}
$$

inequality being strict if

$$||y+\eta_1 x|| \neq ||y+\eta_2 x|| \quad \text{i.e. when}$$

$$\phi(\eta_1) \neq \phi(\eta_2).$$

That proves the inequality (2.1.3).

Now suppose $0 < \alpha < 1$. We have then by (2.1.1) to (2.1.3)

(2.1.4) $\qquad \phi(\alpha) < \alpha\phi(1) + (1-\alpha)\ \phi(0)$ and

(2.1.5) $\qquad \phi(1) < \alpha\phi(\alpha) + (1-\alpha)\ \phi(\alpha+1)$

$$= \alpha\phi(\alpha) + (1-\alpha)\ [\phi(\alpha) + \phi(1) - \phi(0)]$$

and that gives

$$\alpha\phi(1) + (1-\alpha)\ \phi(0) < \phi(\alpha)$$

contradicting (2.1.4).

In case $\alpha > 1$, we use convexity of ϕ and (2.1.1) and (2.1.2) to obtain the fact that

$$\phi(0) \neq \phi(\alpha)$$

and $\quad \phi(1) \neq \phi(\alpha+1).$

Using (2.1.3) again we obtain

(2.1.6) $$\phi(1) < \frac{\alpha-1}{\alpha}\ \phi(0) + \frac{1}{\alpha}\ \phi(\alpha)$$

(2.1.7) $$\begin{aligned}\phi(\alpha) &< \frac{1}{\alpha}\ \phi(1) + \left(\frac{\alpha-1}{\alpha}\right)\ \phi(\alpha+1)\\ &= \frac{\phi(1)}{\alpha} + \left(\frac{\alpha-1}{\alpha}\right)\ [\phi(\alpha) + \phi(1) - \phi(0)]\end{aligned}$$

which yields

$$\phi(\alpha) < \alpha\phi(1) - \alpha\phi(0) + \phi(0)$$

contradicting (2.1.6).

In case $\alpha = 1$, we have

$$\phi(2) = \phi(1) + ||x||^2 = \phi(0) + 2||x||^2 \text{ so that}$$

$$\begin{aligned}\phi(1) &< \frac{1}{2}\ [\phi(0) + \phi(2)]\\ &= \phi(0) + ||x||^2\end{aligned}$$

which is false. Thus always we get a contradiction. Therefore Pythagorean orthogonality is unique in any normed linear space.

Our next theorem combines two results of Day (Theorems 5.1 and 5.2 [7]) characterizing inner-product spaces. Day first proves that a normed linear space B is uniformly convex with a modulus of convexity $\delta(\epsilon) \geq \delta_2(\epsilon) = 1-(1-(\epsilon/2)^2)^{1/2}$ for $0 < \epsilon \leq 2$ if and only if B is an inner-product space. We give a proof which is, perhaps, simpler.

Theorem 2.1.3. Let X be a normed linear space. Then the following are equivalent :

(i) $x, y \in X$, $||x+y||^2 = ||x||^2 + ||y||^2 \Longrightarrow ||x+y|| = ||x-y||$.

(ii) $x, y \in X$, $||x+y|| = ||x-y|| \Longrightarrow ||x+y||^2 = ||x||^2 + ||y||^2$.

(iii) X is an inner-product space.

Proof. Let us first show that if (i) holds, then X is strictly convex. If not, then there exist points $x, y \in X$, $x \neq y$ such that

$$||x|| = ||y|| = ||\tfrac{x+y}{2}|| = 1.$$

Clearly $\frac{x+y}{2} \not\perp_P y$. By Theorem 1.1.3 there exists $\alpha \neq 0$ such that

$$\frac{x+y}{2} \perp_P \alpha(\tfrac{x+y}{2}) + y \text{ i.e.}$$

$$(2.1.8) \qquad ||(1+\alpha)\tfrac{x+y}{2} + y||^2 = 1 + ||\alpha(\tfrac{x+y}{2}) + y||^2.$$

But then (i) implies that

$$(2.1.9) \qquad ||(1+\alpha)(\tfrac{x+y}{2}) + y|| = ||(1-\alpha)(\tfrac{x+y}{2}) - y||.$$

By Theorem 1.1.7

$$|\alpha| \leq 1.$$

For $|\alpha| \leq 1$, (2.1.8) yields

$$2 + \alpha = (2+\alpha)\ \left|\left|\frac{(1+\alpha)\ (\frac{x+y}{2}) + y}{2 + \alpha}\right|\right|$$

$$= \left|\left|\left(\frac{1-\alpha}{2}\right) x - \left(\frac{1+\alpha}{2}\right) y\right|\right| \leq \frac{1-\alpha}{2} + \frac{1+\alpha}{2} = 1.$$

That means $\alpha \leq -1$. Thus α has to be -1. (2.1.8) then yields

$$1 = ||y||^2 = 1 + ||-x+y||^2$$

i.e. $x = y$, which contradicts the assumption.

Therefore X has to be strictly convex.

To prove (i) ==> (ii), suppose the contrary. Then there exists a pair of points $x, y \in X$ such that

$$||x+y|| = ||x-y|| \quad \text{but}$$
$$||x+y||^2 \neq ||x||^2 + ||y||^2.$$

Choose $\alpha \neq 0$ such that

$$||x+\alpha x+y||^2 = ||x||^2 + ||\alpha x+y||^2.$$

But then by (i)

$$||x+\alpha x+y|| = ||x - (\alpha x+y)||.$$

Thus $x \perp_I y$ and $x \perp_I \alpha x+y$, which is contrary to Lemma 2.1.1. That proves

(i) ==> (ii).

To prove (ii) ==> (iii), let

$$||x|| = ||y|| = 1.$$

Then we have

$$||(x+y) + (x-y)|| = ||(x+y) - (x-y)||$$

and therefore by (ii)

$$||x+y+x-y||^2 = ||x+y||^2 + ||x-y||^2.$$

Thus for $||x|| = ||y|| = 1$,

$$||x+y||^2 + ||x-y||^2 = 4.$$

By the criterion (D_1), we get the result.

(iii) ==> (i) is trivial.

Next theorem shows that Pythagorean orthogonality implying Birkhoff-James orthogonality or the reverse implication characterizes inner-product spaces.

Theorem 2.1.4. Let X be a normed linear space. Then the following are equivalent :

(i) $x, y \in X$, $||x+y||^2 = ||x||^2 + ||y||^2 \Longrightarrow$

$$||x+\lambda y|| \geq ||x|| \text{ for all } \lambda \in R.$$

(ii) $x, y \in X$, $||x+\lambda y|| \geq ||x||$ for all $\lambda \in R \Longrightarrow ||x+y||^2$

$$= ||x||^2 + ||y||^2.$$

(iii) X is an inner-product space.

Proof. First we will show that (i) implies strict convexity of the norm. If not, let

$$||x|| = ||y|| = ||\tfrac{x+y}{2}|| = 1, \; x \neq y.$$

Here $$||x + \tfrac{x+y}{2}||^2 = 4 \neq ||x||^2 + ||\tfrac{x+y}{2}||^2 = 2$$

i.e. $\frac{x+y}{2} \not\perp_P x.$

Choose α such that

$$\tfrac{x+y}{2} \perp_P \alpha \tfrac{x+y}{2} + x.$$

Then

$$(2.1.10) \qquad ||(\alpha+1) \tfrac{x+y}{2} + x||^2 = 1 + ||\alpha \tfrac{(x+y)}{2} + x||^2;$$

but then (i) implies that

$$(2.1.11) \qquad ||\tfrac{x+y}{2} + \alpha.\lambda(\tfrac{x+y}{2}) + \lambda x|| \geq ||\tfrac{x+y}{2}|| = 1 \quad \text{for all } \lambda .$$

Choose $\lambda = -\frac{1}{\alpha}$. (2.1.11) gives

$$|\alpha| \leq 1.$$

Writing (2.1.11) as follows

$$(2.1.12) \qquad ||(\tfrac{1}{2} + \tfrac{\alpha\lambda}{2} + \lambda)x + (\tfrac{1}{2} + \tfrac{\alpha\lambda}{2})y|| \geq 1 \quad \text{for all } \lambda$$

and putting $\lambda = -\frac{1}{\alpha+2}$, we get

$$|\tfrac{1}{2} - \tfrac{\alpha}{2(\alpha+2)}| \geq 1$$

or $\left|\frac{1}{\alpha+2}\right| \geq 1$, which combined with $|\alpha| \leq 1$ yields $\alpha = -1$. That means

$$\frac{x+y}{2} \perp_P \frac{x-y}{2} .$$

Therefore

$$||\frac{x+y}{2} + \frac{x-y}{2}||^2 = ||\frac{x+y}{2}||^2 + ||\frac{x-y}{2}||^2$$

$$\text{or} \qquad 1 \qquad = \qquad 1 \quad + ||\frac{x-y}{2}||^2 .$$

Hence $x = y$, which contradicts the supposition. So X is strictly convex.

Now we show that (i) ==> (ii). Suppose that

$$||x+\lambda y|| \geq ||x|| \text{ for all } \lambda \text{ but } ||x+y||^2 \neq ||x||^2 + ||y||^2.$$

By Theorem 1.1.3, there exists $\alpha \neq 0$ such that

$$y \perp_P \alpha y + x.$$

Then $$\alpha y + x \perp_P y.$$

Now (i) gives $\alpha y+x \perp_J y$, but that contradicts left uniqueness of Birkhoff-James orthogonality in a strictly convex space. So (i) ==> (ii) is proved.

To prove (ii) ==> (iii), we first prove the following lemma, a special case of which was proved by Sundaresan [43] by using a geometrical argument. Our proof is analytical.

<u>Lemma 2.1.5.</u> Let X be a normed linear space and $x(\neq 0) \in X$ and $p > 0$. Then there exists a $y \in X$ such that

$$x \perp_J y \text{ and } x+y \perp_J x-py.$$

Proof. Let $x \neq 0$ be given. Choose any $y \neq 0$ such that $x \perp_J y$. By Theorem 1.1.8, we will have

$q_+'(x,y) \geq 0$ and $q_-'(x,y) \leq 0$, where $q_+'(x,y)$ and $q_-'(x,y)$ are respectively the right and the left derivatives of the norm at x in the direction of y.

Put

$$g(\lambda) = ||x + \lambda y||.$$

It is easily seen that $g(\lambda)$ is a continuous convex function of λ and

$$g_+'(\lambda) = \lim_{t \to 0^+} \frac{g(\lambda+t) - g(\lambda)}{t} = \lim_{t \to 0^+} \frac{||(x+\lambda y)+ty||-||x+\lambda y||}{t}$$

$$= q_+'(x+\lambda y,y)$$

similarily

$$g_-'(\lambda) = q_-'(x+\lambda y,y).$$

Here we observe that the right derivarive $g_+'(\lambda)$ of the continuous function $g(\lambda)$ is a nondecreasing function continuous from the right and the left derivative $g_-'(\lambda)$ of the continuous function $g(\lambda)$ is also a nondecreasing function continuous from the left i.e.

$$\lim_{\lambda \to \lambda_0^+} g_+'(\lambda) = g_+'(\lambda_0)$$

and

$$\lim_{\lambda \to \lambda_0^-} g_-'(\lambda) = g_-'(\lambda_0)$$

(see for example Krasnosel'skii and Rutickii [Lemma 1.2,28]).

Using Theorem 1.1.2, we have for $\lambda \leq 0$,

$$\begin{aligned} q'_+(x+\lambda y, x-p\lambda y) &= q'_+(x+\lambda y, x+\lambda y-(1+p)\lambda y) \\ &= ||x+\lambda y|| + q'_+(x+\lambda y, -(1+p)\lambda y) \\ &= ||x+\lambda y|| - (1+p)\lambda\, q'_+(x+\lambda y, y) \\ &= ||x+\lambda y|| - (1+p)\lambda\, g'_+(\lambda) \end{aligned}$$

and for $\lambda \geq 0$

$$\begin{aligned} q'_+(x+\lambda y, x-p\lambda y) &= ||x+\lambda y|| + (1+p)\lambda\, q'_+(x+\lambda y, -y) \\ &= ||x+\lambda y|| - (1+p)\lambda\, q'_-(x+\lambda y, y) \\ &= ||x+\lambda y|| - (1+p)\lambda\, g'_-(\lambda) \end{aligned}$$

So the function $H(\lambda) = q'_+(x+\lambda y, x-p\lambda y)$ is defined for all λ and is such that

for $\lambda_o > 0$

$$\lim_{\lambda \to \lambda_o^-} H(\lambda) = ||x + \lambda_o y|| - (1+p)\, \lambda_o\, g'_-(\lambda_o) = H(\lambda_o)$$

and for $\lambda_o < 0$

$$\lim_{\lambda \to \lambda_o^+} H(\lambda) = ||x + \lambda_o y|| - (1+p)\, \lambda_o\, g'_+(\lambda_o) = H(\lambda_o)$$

Now set

$$G(\lambda) = q'_-(x+\lambda y,\ x-p\lambda y), \quad -\infty < \lambda < \infty.$$

We see that

$$G(\lambda) = -q'_+(x+\lambda y,\ -x+p\lambda y)$$

$$= -\left[q'_+(x+\lambda y,\ -x-\lambda y+\lambda y+p\lambda y) \right]$$

$$= -\left[-||x+\lambda y|| + q'_+(x+\lambda y, \lambda(p+1)y) \right]$$

$$= ||x+\lambda y|| - q'_+(x+\lambda y,\ (p+1)\ \lambda y)$$

Hence for $\lambda \geq 0$

$$G(\lambda) = ||x+\lambda y|| - (1+p)\lambda\ q'_+(x+\lambda y, y)$$

$$= ||x+\lambda y|| - (1+p)\ \lambda g'_+(\lambda)$$

and for $\lambda \leq 0$

$$G(\lambda) = ||x+\lambda y|| - \lambda(1+p)\ q'_-(x+\lambda y, y)$$

$$= ||x+\lambda y|| - \lambda(1+p)\ g'_-(\lambda)$$

From these equations we obtain that

for $\lambda_0 > 0$

$$\lim_{\lambda \to \lambda_0^+} G(\lambda) = ||x+\lambda_0 y|| - (1+p)\ \lambda_0\ g'_+(\lambda_0) = G(\lambda_0),$$

and for $\lambda_0 < 0$

$$\lim_{\lambda \to \lambda_0^-} G(\lambda) = ||x+\lambda_0 y|| - (1+p)\ \lambda_0\ g'_-(\lambda_0) = G(\lambda_0).$$

Now $H(0) = ||x|| > 0,$

and

$$\begin{aligned} H(\lambda) &= q_+'(x+\lambda y, x-p\lambda y) \\ &= q_+'(x+\lambda y, -px-p\lambda y+x+px) \\ &= q_+'(x+\lambda y, -p(x+\lambda y) + (p+1)x) \\ &= -p \, ||x+\lambda y|| + q_+'(x+\lambda y, (p+1)x). \end{aligned}$$

Therefore

$$H(\lambda) \to -\infty \quad \text{as } \lambda \to \pm\infty.$$

Suppose

$$a = \sup \{\lambda \geq 0 : H(\lambda) > 0\} .$$

Clearly $H(a) \geq 0$, because $H(\lambda)$ is continuous from the left for $\lambda > 0$.

Suppose $G(a) > 0$. Then $G(\lambda) > 0$ for some values of $\lambda > a$, because $G(\lambda)$ is continuous from the right for $\lambda > 0$.
But

$$H(\lambda) \geq G(\lambda) \quad \text{for } \lambda \geq 0.$$

So a cannot be the supremum which is a contradiction.

Thus $G(a) \leq 0$.

That means

$$q_+'(x+ay, x-pay) \geq 0$$

$$\text{and} \quad q_-'(x+ay, x-pay) \leq 0.$$

This shows that

$$x + ay \perp_J x-pay$$

by Theorem 1.1.8.

Replacing ay by y, we complete the proof of the Lemma.

Remark 2.1.6. When p = 1 in Lemma 2.1.5, we have the special case proved by Sundaresan (Lemma 1, [43]).

We now complete the proof of the (ii) ==> (iii) part of the theorem. Let

$$||x|| = ||y|| = 1$$

be given. If $x \perp_J y$ and $x+y \perp_J x-y$, we have

$$4 = ||x+y+x-y||^2 = ||x+y||^2 + ||x-y||^2.$$

If $x \not\perp_J y$, by the lemma we can choose an element $z \in X$ such that

$$x \perp_J z \text{ and } x+z \perp_J x-z.$$

Then

$$||x||^2 = ||\frac{(x+z) + (x-z)}{2}||^2 = ||\frac{x+z}{2}||^2 + ||\frac{x-z}{2}||^2$$

$$= ||\frac{x}{2}||^2 + ||\frac{z}{2}||^2 + ||\frac{x}{2}||^2 + ||\frac{z}{2}||^2$$

which yields $||x|| = ||z|| = 1$.

Let α and β be such that $y = \alpha x + \beta z$. Then

$$||y||^2 = ||\alpha x+\beta z||^2 = ||\alpha x||^2 + ||\beta z||^2$$
$$= \alpha^2 + \beta^2 = 1,$$

$$||x+y||^2 = ||(1+\alpha)x+\beta z||^2 = (1+\alpha)^2 + \beta^2 \text{ and}$$

$$||x-y||^2 = ||(1-\alpha)x - \beta z||^2 = (1-\alpha)^2 + \beta^2.$$

Thus

$$||x+y||^2 + ||x-y||^2 = 2(\alpha^2+\beta^2) + 2 = 4.$$

By the criterion (D_1), we obtain the result.

Lorch [31] proved the criterion (L) mentioned in Chapter I which was a considerably weakened form of the criterion (F') of homogeneity of isosceles orthogonality proved by James. The following corollary is analogus to the above in the context of Pythagorean orthogonality.

<u>Corollary 2.1.7.</u> Let X be a normed linear space. Then X is an inner-product space if and only if the folllwing holds :

(L') for some $\gamma \neq 0,1$, $||x+y||^2 = ||x||^2 + ||y||^2 \Longrightarrow$

$$||x+\gamma y||^2 = ||x||^2 + ||\gamma y||^2.$$

Proof. The case of $\gamma = -1$ has been proved by Day [7]. Without loss of generality it can be assumed to be greater than 1. Now suppose x and y $\in$ X such that

$$||x+y||^2 = ||x||^2 + ||y||^2.$$

Then by repeated application of (L'), we will have for $n \geq 1$

$$||y+\gamma^n x||^2 = ||y||^2 + ||\gamma^n x||^2,$$

or

$$\frac{||x + \frac{1}{\gamma^n} y||^2 - ||x||^2}{\frac{1}{\gamma^n}} = \frac{||y||^2}{\gamma^n}.$$

Therefore for all $n \geq 1$

$$(||x + \frac{1}{\gamma^n} y|| + ||x||)\left(\frac{||x + \frac{1}{\gamma^n} y|| - ||x||}{\frac{1}{\gamma^n}}\right) = \frac{||y||^2}{\gamma^n}.$$

In the limit as $n \to \infty$, we have

$$2||x||\, q'_+(x,y) = 0.$$

But in view of Theorem 1.1.7, it means that $x \perp_J y$; therefore by Theorem 2.1.4, X must be an inner-product space.

If X is an inner-product space, then (L') holds any way.

To complete the picture we have the following :

<u>Theorem 2.1.8.</u> In a normed linear space X, each one of the following conditions is necessary as well as sufficient for X to be an inner-product space.

(i) $x \perp_I y \implies x \perp_R y$.

(ii) $x \perp_J y \implies x \perp_R y$.

(iii) $x \perp_P y \implies x \perp_R y$.

(iv) $x \perp_J y \implies x \perp_I y$.

(v) $x \perp_I y \implies x \perp_J y$.

Proof. If any one of the first three holds, then by Theorems 1.1.3 and 1.1.4 the result follows.

Suppose (iv) holds. Let $x \neq 0$, $y \in X$. There exists α such that $x \perp_J \alpha x + y$. But then $x \perp_J k(\alpha x+y)$, because

Birkhoff-James orthogonality is homogeneous. By (iv)

$x \perp_I k(\alpha x+y)$ for every k. But then $x \perp_R \alpha x + y$, and

Roberts' orthogonality is nontrivial. Therefore by Theorem 1.1.4, X is an inner-product space.

To prove the sufficiency of (v) we proceed as follows :

Let $||x|| = ||y||$. Then $x + y \perp_I x-y$ and therefore $x+y \perp_J x-y$. Thus

$$||x+y + \lambda(x-y)|| \geq ||x+y|| \quad \text{for all } \lambda \in R.$$

In particular let $\lambda = \frac{\alpha^2-1}{\alpha^2+1}$ where $\alpha > 1$. Then

$$||(x+y) + \frac{\alpha^2-1}{\alpha^2+1}(x-y)|| \geq ||x+y||.$$

Therefore for $\alpha > 1$

$$||\alpha x + \alpha^{-1}y|| \geq \frac{\alpha^2+1}{2\alpha}||x+y|| \geq ||x+y||.$$

Thus (v) implies the criterion

(L_5) $\quad ||x|| = ||y|| \Longrightarrow ||\alpha x + \alpha^{-1}y|| \geq ||x+y||$ for all $\alpha > 1$

for inner-product spaces proved by Lorch [31].

Remark 2.1.9. (a) A proof of (v) appears in Day [6] where it is listed as criterion (M).

(b) In a research announcement, Holub [18] has stated (iv) and (v) of the above Theorem without proof - (v) in the equivalent form

$$||x|| = ||y|| \implies x+y \perp_J x-y.$$

2.2 A Theorem of plane geometry involving Orthogonality and Characterization of Inner-Product Spaces.

The next couple of characterizations of inner-product spaces obtained by us in the thesis are modeled on the following proposition in plane geometry :

" Suppose ΔABC is a right-angled triangle, right-angled at A and points P and Q are on AB and AC respectively. Then

$$BQ^2 + PC^2 = BC^2 + PQ^2. \text{ "}$$

The corresponding proposition in an inner-product space is that for x and y in an inner-product space X

$$(x,y) = 0 \iff ||ax+y||^2 + ||x+by||^2 = ||ax+by||^2 + ||x+y||^2$$

for $0 < a,\ b < 1$.

In an inner-product space all orthogonalities - isosceles, Pythagorean and Birkhoff-James - are equivalent to the inner product orthogonality i.e.

$$x \perp y \iff (x,y) = 0.$$

In view of this the following results are quite interesting.

Theorem 2.2.1. Let X be a normed linear space and $a,b \in R$ such that $0 < a,\ b < 1$. Then the following are equivalent :

(i) $\quad x, y \in X$ and $||x+y||^2 + ||ax+by||^2 = ||ax+y||^2 +$

$$+ ||by+x||^2 \Longrightarrow x \perp_J y.$$

(ii) $\quad x, y \in X$ and $x \perp_J y \Longrightarrow ||x+y||^2 + ||bx+ay||^2 = ||bx+y||^2 +$

$$+ ||x+ay||^2$$

(iii) X is an inner-product space.

Proof. Let us first prove that if (i) holds, then X is strictly convex. If not, choose x and $y \in X$ such that

$$||x|| = ||y|| = ||\tfrac{x+y}{2}|| = 1$$

and such that x and y are extreme points of the unit ball of X. Clearly

$$||\tfrac{x+y}{2} + y||^2 + ||a(\tfrac{x+y}{2}) + by||^2 \neq ||a(\tfrac{x+y}{2})+y||^2 + ||\tfrac{x+y}{2} + by||^2$$

for otherwise

$4 + (a+b)^2 = (a+1)^2 + (b+1)^2$, which requires $a = 1$ or $b = 1$. Let $a \geq b$ without loss of generality.
We choose $\alpha \neq 0$ such that

$$||\tfrac{x+y}{2} + \alpha(\tfrac{x+y}{2}) + y||^2 + ||a\,\tfrac{x+y}{2} + b(\alpha\,\tfrac{x+y}{2} + y)||^2$$

$$(2.2.1) \qquad = ||a\,\tfrac{x+y}{2} + \alpha\,\tfrac{x+y}{2} + y||^2 + ||\tfrac{x+y}{2} + b\,\alpha\,\tfrac{(x+y)}{2} + by||^2.$$

That such a choice of α is possible will be shown in the

Lemma 2.2.2. Let X be a normed linear space and $0 < a, b < 1$, $0 \neq x, y \in X$. Then there exists a number α such that

$$||(\alpha+1)x+y||^2+||ax+b(\alpha x+y)||^2=||ax+(\alpha x+y)||^2+||x+b(\alpha x+y)||^2$$

Proof of Lemma 2.2.2. Set

$$g(t) = ||x+tx+y||^2+||ax+btx+by||^2-||ax+tx+y||^2-||x+btx+by||^2$$

$$= t^2 \left[||x + \tfrac{x+y}{t}||^2+||bx+ \tfrac{ax+by}{t}||^2 - ||x + \tfrac{ax+y}{t}||^2 - ||bx + \tfrac{x+by}{t}||^2 \right]$$

$$= t^2 \left[(||x + \tfrac{x+y}{t}||^2-||x||^2) + (||bx + \tfrac{ax+by}{t}||^2 - ||bx||^2) - (||x + \tfrac{ax+y}{t}||^2 - ||x||^2) - (||bx + \tfrac{x+by}{t}||^2 - ||bx||^2) \right].$$

Thus for $t \neq 0$

$$\frac{g(t)}{t} = \frac{||x + \frac{x+y}{t}||^2 - ||x||^2}{\frac{1}{t}} + \frac{||bx + \frac{ax+by}{t}||^2 - ||bx||^2}{\frac{1}{t}} - \frac{||x + \frac{ax+y}{t}||^2 - ||x||^2}{\frac{1}{t}} - \frac{||bx + \frac{x+by}{t}||^2 - ||bx||^2}{\frac{1}{t}}$$

or

$$\lim_{t \to \infty} \frac{g(t)}{t} = 2||x||q_+'(x,x+y)+2||bx||\, q_+'(bx,ax+by) - 2||x||\, q_+'(x,ax+y) - 2||bx||\, q_+'(bx, x+by)$$

$$= 2\Big[||x||^2+||x||q_+'(x,y)+ab||x||^2+b^2||x||q_+'(x,y) -a||x||^2 - ||x||\, q_+'(x,y)-b||x||^2 - b^2||x||\, q_+'(x,y)\Big]$$

$$= 2\ ||x||^2\ (1 + ab - a - b)$$

$$= 2\ ||x||^2\ (1-a)\ (1-b) > 0.$$

Similarly

$$\lim_{t \to -\infty} \frac{g(t)}{t} = 2\ ||x||^2\ (1-a)\ (1-b) > 0.$$

Therefore as $t \to \infty$, $g(t) \to \infty$ and

as $t \to -\infty$, $g(t) \to -\infty$.

Hence there exists a number α such that

$g(\alpha) = 0$, which was to be proved.

Continuing with the proof of the Theorem 2.2.1, we have from the equation (2.2.1) and the hypothesis (i)

$$\frac{x+y}{2} \perp_J \alpha\left(\frac{x+y}{2}\right) + y.$$

Thus

$$(2.2.2) \qquad \left|\left|\frac{x+y}{2} + k\ \alpha\frac{x+y}{2} + ky\right|\right| \geq \left|\left|\frac{x+y}{2}\right|\right| = 1 \text{ for all } k \in R.$$

Putting $k = -\frac{1}{\alpha}$ we obtain from (2.2.2) that

$$|\alpha| \leq 1.$$

Now rewriting (2.2.2) in the form

$$(2.2.3) \qquad \left|\left|\left(\frac{1}{2} + \frac{\alpha k}{2}\right) x + \left(\frac{1}{2} + \frac{k\alpha}{2} + k\right)y\right|\right| \geq 1 \text{ for all } k.$$

Choose k such that $\frac{1}{2} + \frac{k\alpha}{2} + k = 0$. Then from (2.2.3) we have

$$|\alpha+2| \leq 1.$$

Now $|\alpha| \leq 1$ and $|\alpha+2| \leq 1$ imply $\alpha = -1$. Putting $\alpha = -1$ in (2.2.1) yields

$$1 + ||(a-b)(\tfrac{x+y}{2})+by||^2 = ||(a-1)(\tfrac{x+y}{2}) + y||^2$$
$$+ ||\tfrac{x+y}{2}(1-b) + by||^2$$

or

$$1 + ||\tfrac{a-b}{2} x + \tfrac{a+b}{2} y||^2 = ||\tfrac{a-1}{2} x + \tfrac{a+1}{2} y||^2 + ||\tfrac{1-b}{2} x + \tfrac{1+b}{2} y||^2$$

and that gives

$$1 + a^2 = ||\tfrac{a-1}{2} x + \tfrac{a+1}{2} y||^2 + 1$$

or
$$||\tfrac{a-1}{2a} x + \tfrac{a+1}{2a} y|| = 1.$$

Writing

$$y = (\tfrac{1-a}{1+a}) x + (1 - \tfrac{1-a}{1+a}) (\tfrac{a-1}{2a} x + \tfrac{a+1}{2a} y),$$

we see that y is a convex combination of two points of the unit sphere which is not possible, since y was taken to be an extreme point of the unit ball and therefore we meet a contradiction. Thus X must be strictly convex if (i) holds. The case of $b \geq a$ is similarly dealt with.

Now we prove that (i) implies (ii). If not, let $x \perp_J y$ and

$$||x+y||^2 + ||bx+ay||^2 \neq ||bx+y||^2 + ||x+ay||^2.$$

There exists $\alpha \neq 0$ such that

$$||\alpha y+x+y||^2+||a(\alpha y+x)+by||^2=||a(\alpha y+x)+y||^2+||\alpha y+x+by||^2$$

but then (i) implies that

$$\alpha y + x \perp_J y,$$

which violates the left **uniqueness** of Birkhoff-James orthogonality in the strictly convex space X (see Theorem 1.1.6). Hence (i) implies (ii).

Finally to prove (ii) ==> (iii), let $y \perp_J x$. By (ii) and the homogeneity of the orthogonality

$$\begin{aligned}
||y+x||^2 &= ||by+x||^2 + ||y+ax||^2 - ||by+ax||^2 \\
&= (||b^2y+x||^2 + ||by+ax||^2 - ||b^2y+ax||^2) \\
&\quad + (||by+ax||^2 + ||y+a^2x||^2 - ||by+a^2x||^2) \\
&\quad - ||by+ax||^2 \\
&= (||b^2y+x||^2 + ||y+a^2x||^2) - ||b^2y+ax||^2 \\
&\quad - ||by+a^2x||^2 + ||by+ax||^2 \\
&= (||b^2y+x||^2+||y+a^2x||^2 -||b^2y+ax||^2-||by+a^2x||^2) \\
&\quad + (||b^2y+ax||^2 + ||by+a^2x||^2 - ||b^2y + a^2x||^2) \\
&= ||b^2y+x||^2 + ||y+a^2x||^2 - ||b^2y + a^2x||^2.
\end{aligned}$$

By induction, we get

$$y \perp_J x \implies ||y+x||^2 = ||b^n y+x||^2 + ||y+a^n x||^2 - ||b^n y+a^n x||^2$$

for all $n \geq 1$.

In the limit when $n \to \infty$, one gets

$$y \perp_J x \implies ||x+y||^2 = ||x||^2 + ||y||^2.$$

But that is sufficient for X to be an inner-product space (Theorem 2.1.4).

Now to complete the proof of the Theorem, it is trivially seen that (iii) $\implies$ (i).

Theorem 2.2.3. Let X be a normed linear space and $0 < a, b < 1$. Consider the following statements :

(i) $x, y \in X$ and $||x+y||^2 + ||ax+by||^2 = ||ax+y||^2 + ||x+by||^2 \implies$

$$||x+y|| = ||x-y||.$$

(ii) $x, y \in X$ and $||x+y|| = ||x-y|| \implies$

$$||x+y||^2 + ||ax+by||^2 = ||ax+y||^2 + ||x+by||^2$$

(iii) X is an inner-product space.

Then (i) $\implies$ (ii); (ii) $\implies$ (iii) when $a = b$, and (iii) $\implies$ (i).

Proof. As before we prove first that if (i) holds, then X is strictly convex. If not, choose x and y such that they are extreme points of the unit ball of X and

$$||x|| = ||y|| = ||\tfrac{x+y}{2}|| = 1.$$

Assume $a \geq b$. Then

$$||\tfrac{x+y}{2} + y||^2 + ||a\ (\tfrac{x+y}{2}) + by||^2 = 4+(a+b)^2$$

$$\neq (a+1)^2 + (b+1)^2 = ||a(\tfrac{x+y}{2}) + y||^2 + ||\tfrac{x+y}{2} + by||^2$$

As in Theorem 2.2.1,we have an $\alpha \neq 0$ such that

(2.2.4) $$||(1+\alpha)\ \tfrac{x+y}{2} + y||^2 + ||(a+b\alpha)\ (\tfrac{x+y}{2}) + by||^2$$

$$= ||(a+\alpha)\ \tfrac{x+y}{2} + y||^2 + ||(b\alpha+1)\ (\tfrac{x+y}{2})+by||^2$$

But then by (i) we will have

(2.2.5) $$||(1+\alpha)\ \tfrac{x+y}{2} + y|| = ||(\alpha-1)\ \tfrac{x+y}{2} + y||$$

$$= ||(\tfrac{\alpha}{2} - \tfrac{1}{2})\ x + (\tfrac{\alpha}{2} + \tfrac{1}{2})y||$$

From equation (2.2.5) and Theorem 1.1.7 one obtains

$$|\alpha| \leq 1.$$

If $0 < \alpha \leq 1$, then the equation (2.2.4) yields

$$(2+\alpha)^2 + [a+b(\alpha+1)]^2 = (1+a+\alpha)^2 + (b\alpha+b+1)^2$$

or $\alpha = -1$

which is false.

If $-1 \leq \alpha < 0$, then from (2.2.5), we have

$$(2+\alpha) \leq (\tfrac{1}{2} + \tfrac{\alpha}{2}) + (\tfrac{1}{2} - \tfrac{\alpha}{2}) = 1\ ,$$

or $\alpha \leq -1$. Thus $\alpha = -1$ is the only possible value of (2.2.4). Then the equation (2.2.4) yields as in the previous theorem

$$1 = ||\frac{a-1}{2a} x + \frac{1+a}{2a} y||$$

and

$$y = \frac{1-a}{1+a} x + (1 - \frac{1-a}{1+a}) (\frac{1+a}{2a} y + \frac{a-1}{2a} x)$$

becomes a convex combination of two points on the unit sphere of X. But y was taken to be an extreme point of the unit ball. Thus there is a contradiction. Therefore X is strictly convex.

Now to prove (i) ==> (ii) suppose that (i) does not imply (ii). Then there exist points x and y such that

$$||x+y|| = ||x-y|| \quad \text{and}$$

$$||x+y||^2 + ||ax+by||^2 \neq ||ax + y||^2 + ||x + by||^2.$$

Choose $\alpha \neq 0$ such that

$$||(\alpha+1)x+y||^2+||ax+b(\alpha x+y)||^2=||ax+\alpha x+y||^2+||x+b(\alpha x+y)||^2$$

Then (i) implies

$$||x + \alpha x + y|| = ||x - (\alpha x+y)||$$

Thus we have $x \perp_I y$ and $x \perp_I \alpha x+y$, but that contradicts the uniqueness of isosceles orthogonality in strictly convex spaces proved in Lemma 2.1.1. Hence (i) ==> (ii).

Now suppose (ii) holds with a = b. Let

$$||x|| = ||y|| = 1.$$

Then $||(x+y) + (x-y)|| = ||(x+y) - (x-y)||$

Therefore by (ii) we have

$$||x+y+x-y||^2 + a^2 \, ||x+y+x-y||^2 = ||a(x+y)+(x-y)||^2 + ||x+y+a(x-y)||^2$$

Or

$$||\tfrac{1+a}{2} x - \tfrac{(1-a)}{2} y||^2 + ||\tfrac{1+a}{2} x + \tfrac{1-a}{2} y||^2 = 1 + a^2$$

The result now follows by Day's characterization $(D, \dot{\sim})$ with $\lambda = \mu = \frac{1+a}{2}$ and $\dot{\sim}$ replaced by equality (=).

(iii) ==> (i) is trivial.

<u>Remark 2.2.4.</u> In Theorem 2.2.3 we were not able to prove that (ii) ==> (iii) when $a \neq b$. It also remains to be seen whether the condition

$$x \perp_P y \implies ||ax+by||^2+||x+y||^2 = ||ax+y||^2+||x+by||^2$$

even with $a = b$ characterizes inner product spaces or not.

<u>Theorem 2.2.5.</u> Let X be a normed linear space and p_1, q_1, p_2 and q_2 be real numbers such that

$$p_1 q_1 + p_2 q_2 \neq 0.$$

Then X is an inner-product space if and only if

$$(*) \quad x \perp_J y \implies p_1 \, ||x+q_1 y||^2 + p_2 \, ||x+q_2 y||^2 = p_1 ||x-q_1 y||^2 + p_2 \, ||x-q_2 y||^2.$$

Proof. If Xis an inner-product space, then it is easily verified that (*) must hold.

Suppose now (*) holds with p_1 or $p_2 = 0$ or with $q_1 = q_2$. Then (*) is equivalent to saying that

$$x \perp_J y \implies x \perp_I y$$

and that characterizes inner-product space. If p_1 and p_2 both are different from zero, then (*) can be written as

$$x \perp_J y \implies ||x+q_1y||^2-||x-q_2y||^2=p_2'||x-q_1y||^2 - p_2'||x+q_2y||^2$$

$$\text{with } |p_2'| \leq 1.$$

Or using homogeneity of Birkhoff-James orthogonality, the following holds

$$(**) \quad x \perp_J y \implies ||x+y||^2-||x-y||^2 = a[||x-by||^2-||x+by||^2]$$

$$\text{with } |a| \leq 1 \text{ and } |b| < 1.$$

By induction one can prove that for all $n \geq 1$

$$x \perp_J y \implies ||x+y||^2-||x-y||^2 = a^n [||x+b^ny||^2-||x-b^ny||^2].$$

In the limit as $n \to \infty$, we have

$$x \perp_J y \implies x \perp_I y$$

and hence Xis an inner product space.

Using the same idea we have following

Theorem 2.2.6. Let X be a normed linear space and $p_i \neq 0 \neq q_i$, $i = 1,2,\dots,n$ numbers such that

$$p_i\, q_i > \sum_{j\neq i} p_j\, q_j \quad \text{for some } i.$$

Then X is an inner-product space if and only if the following holds:

$$(*) \qquad x \perp_J y \Longrightarrow \sum_{i=1}^{n} p_i\, ||x+q_i y||^2 = \sum_{i=1}^{n} p_i\, ||x-q_i y||^2.$$

Proof. The (*) is necessary, is easy. To prove sufficiency assume without loss of generality that

$$p_1\, q_1 > \sum_{i=2}^{n} p_i q_i.$$

Let $x \perp_J y$. Then we have

$$||x+q_1y||^2 - ||x-q_1y||^2 = \sum_{i=2}^{n} \frac{p_i}{p_1} [\,||x-q_iy||^2 - ||x+q_iy||^2\,]$$

or

$$||x+y||^2 - ||x-y||^2 = \sum_{i=2}^{n} \frac{p_i}{p_1} [\,||x - \frac{q_i}{q_1} y||^2 - ||x + \frac{q_i}{q_1}||^2\,]$$

$$= \sum_{i=2}^{n} a_i\, [\,||x-b_iy||^2 - ||x+b_iy||^2\,]$$

where $a_i = \frac{p_i}{p_1}$, $b_i = \frac{q_i}{q_1}$, $i = 2,3,\dots,n$.

By using the fact that Birkhoff-James orthogonality is homogeneous we get

$$||x+y||^2-||x-y||^2 = \sum_{i=2}^{n} a_i \sum_{j=2}^{n} a_j[||x+b_jb_iy||^2-||x-b_jb_iy||^2]$$

$$= a_2^2[||x+b_2^2y||^2-||x-b_2^2y||^2]+2a_2a_3[||x+b_2b_3y||^2$$

$$-||x-b_2b_3y||^2]+\ldots+2a_2a_n[||x+b_2b_ny||^2-||x-b_2b_ny||^2]$$

$$+ a_3^2[||x+b_3^2y||^2-||x-b_3^2y||^2]+2a_3a_4[||x+b_3b_4y||^2 -$$

$$- ||x-b_3b_4y||^2]$$

$$+ \ldots + 2a_3a_n[||x+b_3b_ny||^2-||x-b_3b_ny||^2] + \ldots$$

$$\ldots + a_n^2 [||x+b_n^2y||^2 - ||x-b_n^2y||^2].$$

In this way by repeated use of the hypothesis (by induction) we get, for all $m \geq 1$

$$|\ ||x+y||^2-||x-y||^2| = |\sum_{\substack{r_2,r_3\ldots r_n \\ \sum_{i=2} r_i = m}} \binom{m}{r_2\ldots r_n} a_2^{r_2} \ldots a_n^{r_n} .$$

$$\times \left[||x+b_2^{r_2} b_3^{r_3}\ldots b_n^{r_n}y||^2-||x-b_2^{r_2}b_3^{r_3}\ldots b_n^{r_n}y||^2\right]|$$

$$\leq \sum_{\substack{r_2\ldots r_n \\ \Sigma r_i = m}} \binom{m}{r_2r_3\ldots r_n} |a_2b_2|^{r_2} |a_3b_3|^{r_3} \ldots |a_nb_n|^{r_n}$$

$$\times \left|\frac{||x+b_2^{r_2} \ldots b_n^{r_n}y||^2 - ||x-b_2^{r_2} \ldots b_n^{r_n}y||^2}{b_n^{r_2} b_3^{r_3} \ldots b_n^{r_n}}\right| .$$

Let $\phi(\eta) = ||x+\eta y||^2$. Thus $\dfrac{\phi(\eta) - \phi(-\eta)}{\eta}$ is a bounded function of η for $0 < \eta < \infty$. In fact it has a finite limit

as $\eta \to \infty$ and as $\eta \to 0$, and is continuous on $0 < \eta < \infty$.

Let M be a bound for ϕ on $0 < \eta < \infty$. From the above inequality we have for all $m \geq 1$

$$\left| \, ||x+y||^2 - ||x-y||^2 \right| \leq M \sum_{r_2+r_3+\cdots+r_n=m} \binom{m}{r_2 \cdots r_n}$$

$$\times \, |a_2 b_2|^{r_2} \cdots |a_n b_n|^{r_n}$$

$$= M(|a_2 b_2| + |a_3 b_3| + \cdots + |a_n b_n|)^m .$$

Right hand side of this inequality tends to zero as m tends to infinity because $\sum_2^n |a_i b_i| < 1$ from the fact that

$$\sum_{i=2}^{n} |p_i q_i| < p_1 q_1 .$$

Thus in X, $x \perp_J y \implies x \perp_I y$ which is a characterization of inner-product spaces proved in Theorem 2.1.8.

Remark 2.2.7. Define an orthogonality relation in a normed linear space X as follows :

$$x \perp_G y \iff \sum_{i=1}^{\infty} p_i \, ||x+q_i y||^2 = \sum_{i=1}^{\infty} p_i \, ||x-q_i y||^2 .$$

This orthogonality is the so called symmetrical case of Carlsson's orthogonality [4]. Theorem 2.2.6, then says

$$x \perp_J y \implies x \perp_G y$$

implies that X is an inner-product space.

<u>Theorem 2.2.8.</u> Let X be a normed linear space and $a \neq 1$. Then X is an inner-product space if and only if

(A) $\qquad ||x-y|| = ||ax-y|| \implies ||x-y||^2 + a||x||^2 = ||y||^2.$

Proof. Let $x, y \in X$ such that $||x+y|| = ||x-y||$.

Then

$$||\tfrac{2}{1-a}\, y - (-x + \tfrac{a+1}{1-a}\, y)|| = ||\tfrac{2a}{1-a}\, y - (-x + \tfrac{a+1}{1-a}\, y)||.$$

From (A) we get

(B) $\qquad ||x-y||^2 + a||\frac{2y}{1-a} - x||^2 = ||-x + \frac{a+1}{1-a}\, y||^2.$

By applying the same argument to the pair $-x, y$ in place of x and y we obtain

(C) $\qquad ||x+y||^2 + a\, ||\frac{2}{1-a}\, x||^2 = ||x + \frac{a+1}{1-a}\, y||^2.$

From (B) and (C) we get

$$||x + \tfrac{a+1}{1-a}\, y|| = ||x - \tfrac{a+1}{1-a}\, y||.$$

Hence by Lorch's criterion (L) we get the result.

<u>Remark 2.2.9.</u> If we put $a = 3$, we obtain the result of Freese [13] which was mentioned by Blumenthal ([2], p. 7) as a new norm postulate.

2.3 Norm Identities Characterizing Inner-Product Spaces

In this section we show that analogues in a normed linear space X of some metric propositions in Euclidean plane give rise to postulates for X to be an inner-product space.

Later on we show that some of them and two recently published characterizations are actually consequences of the Theorem 1.1.14, mentioned in Chapter I.

A proposition in plane geometry : In any triangle the sum of the squares of the sides is three times the sum of the distances of the vertices from the centre of gravity of the triangle.

The metric analogue is :

Theorem 2.3.1. Let X be a normed linear space. Then X is an inner-product space if and only if for all $x, y, z \in X$

$$||y+z-2x||^2 + ||z+x-2y||^2 + ||x+y-2z||^2 = 3\,[\,||y-x||^2+||x-z||^2+||z-y||^2\,]. \tag{2.3.1}$$

Proof. Necessary part is easy. To prove the sufficiency, let u and $v \in X$. Choose x, y and $z \in X$ such that

$$y + z - 2x = \frac{v}{3} - u \quad \text{and}$$

$$z + x - 2y = \frac{v}{3} + u.$$

Then by equation (2.3.1), we have

$$||\frac{v}{3} - u||^2 + ||\frac{v}{3} + u||^2 + \frac{4}{3^2}||v||^2 = \frac{1}{3}[4||u||^2+||v-u||^2+||u+v||^2]$$

Or

$$||v+u||^2+||v-u||^2=3[\,||\frac{v}{3} + u||^2+||\frac{v}{3} - u||^2 + \frac{4}{3^2}||v||^2] - 4||u||^2. \tag{2.3.2}$$

Replacing v by $\frac{v}{3}$ in (2.3.2) and then substituting the value of

$$||\frac{v}{3}+u||^2 + ||\frac{v}{3}-u||^2$$

so obtained in (2.3.2) we have

$$||v+u||^2 + ||v-u||^2 = 3^2 \left[||\frac{v}{3^2}+u||^2 + ||\frac{v}{3^2}-u||^2 \right] + (\frac{4}{3}+\frac{4}{3^2})\,||v||^2 - 4(1+3)\,||u||^2.$$

By induction we get

$$(2.3.3) \qquad ||v+u||^2+||v-u||^2=3^n \left[||\frac{v}{3^n}+u||^2 + ||\frac{v}{3^n}-u||^2 \right] + \frac{4}{3}\left[1+\frac{1}{3}+\dots+\frac{1}{3^{n-1}}\right]||v||^2 - 4\left[1+3+\dots+3^n\right]||u||^2$$

$$= 3^n \left[||\frac{v}{3^n}+u||^2+||\frac{v}{3^n}-u||^2 \right] + \frac{4}{3}\,\frac{(1-\frac{1}{3^n})}{1-\frac{1}{3}}\,||v||^2 - 4\left[1+3+\dots+3^n\right)||u||^2$$

$$= 3^n \left[||\frac{v}{3^n}+u||^2+||\frac{v}{3^n}-u||^2 \right] + \frac{4}{3}\,\frac{(1-\frac{1}{3^n})}{1-\frac{1}{3}}\,||v||^2 - 4\,\frac{(3^n-1)}{3-1}\,||u||^2$$

$$= 3^n \left[||\frac{v}{3^n}+u||^2+||\frac{v}{3^n}-u||^2-2||u||^2 \right] + 2(1-\frac{1}{3^n})\,||v||^2+2||u||^2.$$

Now

$$\lim_{n \to \infty} 3^n \left[||\frac{v}{3^n} + u||^2 + ||\frac{v}{3^n} - u||^2 - 2||u||^2 \right]$$

$$= \lim_{t \to 0^+} t^{-1}\left[||u+tv||^2 - ||u||^2 \right] + \lim_{t \to 0^+} t^{-1}\left[||u-tv||^2 - ||u||^2 \right]$$

$$= 2\,||u||\,\left[q_+'(u,v) - q_-'(u,v) \right].$$

Therefore on allowing n to increase to infinity in (2.3.3) we get

$$||v+u||^2+||v-u||^2=2||u||^2+2||v||^2+2||u||\,\left[q_+'(u,v)-q_-'(u,v)\right].$$

Thus in view of the fact that

$$q_+'(u,v) \geq q_-'(u,v)$$

we have

$$||u|| = ||v|| = 1 \implies ||u+v||^2 + ||u-v||^2 \geq 4$$

and therefore X is an inner-product space by Schoenberg's characterization (S,$\dot{\text{v}}$).

Another geometrical proposition : The sum of the squares on the sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the squares on the straight line which joins the middle points of the diagonals.

<u>Theorem 2.3.2.</u> (Analogue). Let X be a normed linear space. Then X is an inner-product space if and only if for all x,y,z and w $\in$ X

$$(2.3.4) \qquad ||y-x||^2 + ||z-y||^2 + ||w-z||^2 + ||x-w||^2$$
$$= ||z-x||^2+||w-y||^2+4|| \tfrac{x+z}{2} - \tfrac{y+w}{2}||^2.$$

Proof. The necessity is easily verified and for sufficiency let $z = -x$ and $w = -y$ in (2.3.4). This yields the parallelogram law, thereby proving the sufficient part.

Recently Rakestraw [35] has given the following characterization :

Theorem 2.3.3. Let X be a normed linear space. Then X is an inner-product space if and only if

(R) $n \geq 3$, $x_1, x_2, \ldots, x_n \in X$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers such that

$$\sum_{i=1}^{n} \alpha_i = 0,$$

then

$$||\sum_{i=1}^{n} \alpha_i x_i||^2 = -\sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \, ||x_i - x_j||^2.$$

The proof of the sufficiency of this theorem is immediate : for x_1 and $x_2 \in X$

$$\tfrac{1}{4} \, ||x_1+x_2||^2 = ||\tfrac{1}{2} x_1 + \tfrac{1}{2} x_2 + (-1)\, 0||^2$$
$$= \tfrac{1}{2} \, ||x_1||^2 + \tfrac{1}{2} \, ||x_2||^2 - \tfrac{1}{4} \, ||x_1-x_2||^2$$

by (R) and thus (JN) holds.

We will prove the following modified form of the above theorem.

<u>Theorem 2.3.4</u>. Let X be a normed linear space. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ $(n \geq 3)$ be prescribed non-zero real numbers such that

$$\sum_{i=1}^{n} \alpha_i = 0.$$

Then X is an inner-product space if and only if for all $x_1, x_2, \ldots, x_n \in X$,

$$(2.3.5) \qquad \left\| \sum_{i=1}^{n} \alpha_i \, x_i \right\|^2 = - \sum_{1 \leq i < j \leq n} \| x_i - x_j \|^2.$$

Proof. The proof of the necessary part of Theorem 2.3.4 is the same as that of Theorem 2.3.3. Rakestraw has given a quite nice proof which we are including here for the sake of completeness. The proof is by induction on n. Thus if $n = 3$, then

$$\begin{aligned} \left\| \sum_{i=1}^{3} \alpha_i x_i \right\|^2 &= \| \alpha_1 (x_1 - x_3) + \alpha_2 (x_2 - x_3) \|^2 \\ &= \alpha_1^2 \, \| x_1 - x_3 \|^2 + \alpha_1 \alpha_2 (x_1 - x_3, x_2 - x_3) \\ &\quad + \alpha_1 \alpha_2 (x_2 - x_3, x_1 - x_3) + \alpha_2^2 \, \| x_2 - x_3 \|^2. \end{aligned}$$

Upon replacing $x_2 - x_3$ by $x_1 - x_3 + x_2 - x_1$ and

$$x_1 - x_3 \text{ by } x_2 - x_3 + x_1 - x_2$$

in the second and third terms respectively and using additivity of inner-product, we obtain

$$||\sum_1^3 \alpha_i x_i||^2 = \alpha_1^2||x_1-x_3||^2 + \alpha_1\alpha_2||x_1-x_3||^2 + \alpha_1\alpha_2(x_1-x_3, x_2-x_1)$$

$$+\alpha_1\alpha_2||x_2-x_3||^2 + \alpha_1\alpha_2(x_2-x_3, x_1-x_2) + \alpha_2^2||x_2-x_3||^2$$

$$= \alpha_1(\alpha_1+\alpha_2)||x_1-x_3||^2 + \alpha_2(\alpha_1+\alpha_2)||x_2-x_3||^2$$

$$+ \alpha_1\alpha_2\left[(x_1-x_2, x_2-x_1) + (x_2-x_3, x_2-x_1) + (x_2-x_3, x_1-x_2)\right]$$

$$= -\alpha_1\alpha_2||x_1-x_2||^2 - \alpha_1\alpha_3||x_1-x_3||^2 - \alpha_2\alpha_3||x_2-x_3||^2.$$

Now assume that (2.3.5) holds for some $n \geq 3$. Let $x_1, x_2, \ldots, x_{n+1}$ be in X and $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \neq 0$ be real numbers such that

$$\sum_{i=1}^{n+1} \alpha_i = 0.$$

Set $\lambda = -(\alpha_n + \alpha_{n+1}) = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}$. Then

$$||\sum_{i=1}^{n+1} \alpha_i x_i||^2 = \alpha_{n+1}^2 \; ||\sum_{i=1}^{n} \left(\frac{\alpha_i}{-\alpha_{n+1}}\right) x_i - x_{n+1}||^2$$

$$= \alpha_{n+1}^2 \; ||\sum_{i=1}^{n-1} \frac{\alpha_i}{-\alpha_{n+1}} x_i + \frac{\alpha_n x_n}{-\alpha_{n+1}} - x_{n+1}||^2$$

$$= \alpha_{n+1}^2 ||\frac{\lambda}{-\alpha_{n+1}} \left[\sum_{i=1}^{n-1} \frac{\alpha_i}{\lambda} x_i\right] + \frac{\alpha_n x_n}{-\alpha_{n+1}} - x_{n+1}||^2$$

Since

$$\frac{\lambda}{-\alpha_{n+1}} + \frac{\alpha_n}{-\alpha_{n+1}} - 1 = 0, \text{ it follows that}$$

$$||\sum_{i=1}^{n+1} \alpha_i x_i||^2 = \alpha_{n+1}^2 \left[\frac{-\lambda\alpha_n}{\alpha_{n+1}^2} ||\sum_{i=1}^{n-1} \frac{\alpha_i x_i}{\lambda} - x_n||^2\right.$$

$$\left. + \frac{\lambda}{-\alpha_{n+1}} \; ||\sum_{i=1}^{n-1} \frac{\alpha_i x_i}{\lambda} - x_{n+1}||^2 + \frac{\alpha_n}{-\alpha_{n+1}} ||x_n - x_{n+1}||^2\right].$$

By the induction hypothesis, we have

$$\left|\left|\sum_{i=1}^{n+1} \alpha_i x_i\right|\right|^2 = - \lambda\alpha_n \left[\sum_{1 \le i < j \le n-1} \frac{\alpha_i\alpha_j}{\lambda^2} ||x_i-x_j||^2\right]$$

$$- \lambda\alpha_n \left[\sum_{i=1}^{n-1} \frac{\alpha_i}{\lambda} ||x_i-x_n||^2\right]$$

$$+ (-\lambda\alpha_{n+1}) \left[\sum_{1 \le i < j \le n-1}^{-} \frac{\alpha_i\alpha_j}{\lambda^2} ||x_i-x_j||^2\right]$$

$$+ (-\lambda\alpha_{n+1}) \left[\sum_{i=1}^{n-1} \frac{\alpha_i}{\lambda} ||x_i-x_{n+1}||^2\right] -$$

$$- \alpha_n\alpha_{n+1} ||x_n-x_{n+1}||^2.$$

Simplifying we obtain

$$\left|\left|\sum_{i=1}^{n+1} \alpha_i x_i\right|\right|^2 = -\sum_{1 \le i < j \le n+1} \alpha_i\alpha_j ||x_i-x_j||^2.$$

To prove that if (2.3.5) holds for all choices of $x_1, x_2, \ldots, x_n$, then X must be an inner-product space, we suppose, without loss of generality that $\alpha_1 \neq \pm \alpha_2$. Letting $x_3 = x_4 = \ldots = x_n = 0$ in (2.3.5) we obtain that there exist nonzero numbers such that for all x_1 and $x_2 \in X$ the following holds :

$$(2.3.6) \quad ||\alpha_1 x_1+\alpha_2 x_2||^2 = -\alpha_1\alpha_2||x_1-x_2||^2 - \alpha_1||x_1||^2 \sum_{j=3}^{n} \alpha_j$$

$$- \alpha_2 ||x_2||^2 \sum_{j=3}^{n} \alpha_j$$

$$= -\alpha_1\alpha_2||x_1-x_2||^2 + \alpha_1(\alpha_1+\alpha_2)||x_1||^2$$

$$+ \alpha_2(\alpha_1+\alpha_2) ||x_2||^2$$

$$= \alpha_1^2||x_1||^2+\alpha_2^2||x_2||^2+\alpha_1\alpha_2[||x_1-x_2||^2+||x_1||^2+||x_2||^2]$$

Replacing x_2 by $(-x_2)$ in (2.3.6) we obtain

$$(2.3.7) \quad ||\alpha_1x_1-\alpha_2x_2||^2 = \alpha_1^2\ ||x_1||^2 + \alpha_2^2\ ||x_2||^2 + \alpha_1\alpha_2[||x_1+x_2||^2+||x_1||^2+||x_2||^2]$$

From (2.3.6) and (2.3.7) we have the identity

$$||\alpha_1x_1+\alpha_2x_2||^2 - ||\alpha_1x_1-\alpha_2x_2||^2 = \alpha_1\alpha_2[||x_1-x_2||^2-||x_1+x_2||^2]$$

Thus in the normed linear space X, the following holds :

$$||x_1+x_2|| = ||x_1-x_2|| \implies ||\alpha_1x_1+\alpha_2x_2|| = ||\alpha_1x_1-\alpha_2x_2||$$

which is the criterion (L) of Lorch. Hence X must be an inner-product space.

In another recent paper [22] Johnson has proved the following :

Theorem 2.3.5. Let X be a normed linear space such that for some $n \geq 3$

$$\sum_{m=0}^{n} n_{c_m}\ (-1)^m\ ||x+my||^2 = 0 \text{ for all } x,y \in X,$$

then $\sum_{m=0}^{n} n_{c_m}(-1)^m\ ||x+my||^2 = 0$ for all $n \geq 3$

and X is an inner-product space.

Proof. See [22].

In the following we intend to show that the Theorem 1.1.14 of Carlsson [4] given in the first chapter is quite powerful - powerful enough to yield Theorems 2.3.1, 2.3.4 and 2.3.5 easily.

(1) Alternate proof of Theorem 2.3.4.

After getting equation (2.3.6) we can say that for all $x, y \in X$, we have

$$||\alpha_1x_1+\alpha_2x_2||^2+\alpha_1\alpha_2||x_1-x_2||^2-\alpha_1(\alpha_1+\alpha_2)||x_1||^2-\alpha_2(\alpha_1+\alpha_2)||x_2||^2=0.$$

Therefore the conditions of Theorem 1.1.14 are satisfied if we take

$$a_1 = 1,\ a_2 = \alpha_1\alpha_2,\ a_3 = -\alpha_1(\alpha_1+\alpha_2),\ a_4 = -\alpha_2(\alpha_1+\alpha_2)$$
$$b_1 = \alpha_1, b_2 = 1\ ,\ b_3 = 1\ ,\ b_4 = 0$$
$$c_1 = \alpha_2, c_2 = -1\ ,\ c_3 = 0 \text{ and } c_4 = 1.$$

Hence X is an inner-product space.

(2) Alternate proof of Theorem 2.3.1. From equation (2.3.2) we have for all $u, v \in X$

$$||u+v||^2+||u-v||^2-3||u+\tfrac{v}{3}||^2-3||u-\tfrac{v}{3}||^2+4||u||^2-\tfrac{4}{3}||v||^2 = 0.$$

The conditions of the Theorem 1.1.14 are satisfied if we take

$$a_1 = 1\ ,\ a_2 = 1\ ,\ a_3 = -3\ ,\ a_4 = -3,\ a_5 = 4,\ a_6 = -\tfrac{4}{3}$$
$$b_1 = 1\ ,\ b_2 = 1\ ,\ b_3 = 1\ ,\ b_4 = 1\ ,\ b_5 = 1,\ b_6 = 0$$
$$c_1 = 1\ ,\ c_2 = -1\ ,\ c_3 = \tfrac{1}{3}\ ,\ c_4 = -\tfrac{1}{3},\ c_5 = 0 \text{ and } c_6 = 1.$$

Hence X must be an inner-product space.

(3) Alternate proof of Theorem 2.3.5. Take

$$a_\nu = {}^{n}c_\nu \, (-1)^\nu, \; b_\nu = 1, \; c_\nu = \nu, \; \nu = 1,2,\ldots,n.$$ The

requirements of the Theorem 1.1.14 are again satisfied. Therefore X is an inner-product space.

We end this chapter with the

<u>Remark 2.3.6.</u> One starts getting the impression that the analogue of every reasonably general proposition of Euclidean plane would perhaps yield a characterization of inner-product spaces. It is not so. The following counter-example was suggested by Day and others and was worked out by Kelley [26].

<u>Example 2.3.7.</u> In the Minkowski plane of ordered pairs (x_1,x_2) of real numbers with unit circle a regular dodecagon the following norm identity is satisfied

$$x,y \in X, \; ||x|| = ||y|| = 1 = ||x-y|| \Longrightarrow ||x+y|| = \sqrt{3}.$$

Geometrically stated it says that the medians of equilateral triangles of side length ℓ are of length $\frac{\sqrt{3}}{2}\,\ell$ as they are in the Euclidean plane.

CHAPTER III

Orthogonality for Generalized Inner-Product and Characterization of Inner-Product Spaces

The generalized inner-product $\langle x,y\rangle$ in a normed linear space X, is the right Gâteaux derivative of the functional $\frac{1}{2}\,||x||^2$, at x in the direction of y. The orthogonality relation for the generalized inner-product is

$$x \perp_G y \iff \langle x,y\rangle = 0.$$

In this chapter, we obtain a result on the right existence of G-orthogonal pairs in every two-dimensional subspace and show by a counter example,the left existence may not be there. It is proved that a normed linear space is smooth if and only if the Birkhoff-James orthogonality implies the G-orthogonality, and the left uniqueness of the G-orthogonality is a necessary and sufficient condition for the normed linear space to be strictly convex. Using these results we provide a shorter proof of a theorem of James which states that the symmetry of the G-orthogonality implies the symmetry of the Birkhoff-James orthogonality. R.A. Tapia [46] proved that X must be an inner-product space if the generalized inner-product is either symmetric or linear and D. Laugwitz [29] proved that if the dimension X $\geq$ 3 and the orthogonality for the generalized inner-product is symmetric, then X is an inner-product space. We give alternative proofs of these results.

3.1 Generalized Inner-Product Orthogonality

Let X be a real normed linear space. The norm functional will be denoted by $q(x)$ as before. The generalized inner-product of x with y, denoted by $\langle x,y\rangle$ is defined as follows :

$$\langle x,y\rangle = \frac{1}{2} \lim_{t \to 0^+} \frac{||x+ty||^2 - ||x||^2}{t} .$$

It follows that

$$\langle x,y\rangle = ||x||\ q'_+(x,y).$$

We say x is G-orthogonal to y ($x \perp_G y$) if $\langle x,y\rangle = 0$.

Thus

$$x \perp_G y \iff \text{either } x = 0 \text{ or } q'_+(x,y) = 0.$$

For $\lambda > 0$, the relations

$$q'_+(x,\lambda y) = \lambda q'_+(x,y) \quad \text{and}$$

$$q'_+(\lambda x,y) = q'_+(x,y)$$

show that the G-orthogonality is positive homogeneous.

In what follows, we will be using the results of Theorem 1.1.2 and Theorem 1.1.6, without referring to them again and again. We begin by proving the right existence for G-orthogonality.

<u>Theorem 3.1.1.</u> Let x and y be linearly independent elements of a normed linear space X. There exists a unique number b such that

$$x \perp_G bx + y.$$

Proof. Take $b = \dfrac{-q'_+(x,y)}{||x||}$. Then

$q'_+(x, bx+y) = b||x|| + q'_+(x,y) = 0$. Thus $x \perp_G bx + y$.

If there exist $\alpha, \beta (\alpha \neq \beta)$ such that

$$x \perp_G \alpha x + y \text{ and } x \perp_G \beta x + y, \text{ then we have}$$

$\alpha||x|| + q'_+(x,y) = \beta||x|| + q'_+(x,y)$ which in turn gives $\alpha = \beta$, a contradiction proving the uniqueness of 'b' in $x \perp_G bx+y$.

For G-orthogonality there may be no number 'b' such that

$$bx + y \perp_G x,$$

as the following example shows.

<u>Example 3.1.2.</u> Consider R^2 with the norm

$$||(x_1,x_2)|| = |x_1| + |x_2|.$$

Let $x = (1,0)$ and $y = (0,1)$. Then we have

$$q'_+(\eta x+y, x) = \lim_{t \to 0^+} \frac{||(\eta+t,1)|| - ||(\eta,1)||}{t}$$

$$= \lim_{t \to 0^+} \frac{|\eta+t| - |\eta|}{t}$$

$$= 1 \text{ for } \eta \geq 0$$

$$= -1 \text{ for } \eta < 0.$$

Thus $q'_+(\eta x+y, x) \neq 0$ for all η and therefore, there exists no η for which

$$\eta x + y \perp_G x.$$

<u>Theorem 3.1.3</u>. Let X be a normed linear space. X is smooth if and only if

$$x, y \in X \text{ and } x \perp_J y \Longrightarrow x \perp_G y.$$

Proof. If X is smooth, then $x \perp_J y$ if and only if the Gâteaux derivative of the norm at x in the direction of y is zero. Hence the two orthogonalities are the same.

If X is not smooth, then there exist $0 \neq x$ and $y \in X$ such that $x \perp_J y$ and $x \perp_J (x+y)$. The hypothesis then implies that $x \perp_G y$ and $x \perp_G (x+y)$. But that means

$$q'_+(x,y) = 0 \text{ and } q'_+(x,x+y) = ||x|| + q'_+(x,y) = 0,$$

which is false. That completes the proof.

<u>Theorem 3.1.4.</u> Let X be a normed linear space. X is strictly convex if and only if the G-orthogonality is left unique i.e. if and only if

$$\alpha x + y \perp_G x \text{ and } \beta x + y \perp_G x \Longrightarrow \alpha = \beta.$$

Proof. If X is strictly convex and $\alpha x + y \perp_G x$ and $\beta x+y \perp_G x$, then $\alpha x+y \perp_J x$ and $\beta x+y \perp_J x$ and therefore $\alpha = \beta$.

On the otherhand, if X is not strictly convex, then choose y and z such that

$$||y|| = ||z|| = ||(1-t)y+tz|| = 1 \text{ for } 0 \leq t \leq 1.$$

For $0 < \lambda < 1$

$$q'_+(\lambda(z-y)+y, z-y) = \lim_{t \to 0^+} \frac{||(\lambda+t)(z-y)+y||-||\lambda(z-y)+y||}{t}$$

$$= 0.$$

Thus $\lambda x+y \perp_G x$ for $0 < \lambda < 1$ where $x = z-y$ and hence the proof of the theorem is completed.

The following result is the Theorem 3.5 of James [20]. In view of the above results, we are able to give a shorter proof of it.

Theorem 3.1.5. If in a normed linear space X, the G-orthogonality is symmetric, then the Birkhoff-James orthogonality is also symmetric and X is both strictly convex and smooth.

Proof. Suppose x and y are linearly independent elements of X such that

$$\alpha x + y \perp_G x \text{ and } \beta x+y \perp_G x.$$

Then by symmetry of G-orthogonality

$$x \perp_G \alpha x+y \text{ and } x \perp_G \beta x+y. \quad \text{But then}$$

$$\alpha = \beta = \frac{-q'_+(x,y)}{||x||}.$$

Therefore X is strictly convex.

Now suppose that X is not smooth. Then there exist $x, y \in X$ such that $x \perp_J y$ but $x \not\perp_G y$. Choose $b \neq 0$ such that $y \perp_G by+x$. Then $by+x \perp_G y$ and therefore $by + x \perp_J y$ which contradicts the strict convexity of the space. Hence X is

smooth and both the orthogonalities are the same. That gives the result.

Corollary 3.1.6 (Laugwitz [29, Theorem 4]). Let X be a normed linear space of dimension ≥ 3. Then X is an inner-product space if and only if $\langle x,y\rangle = 0$ implies $\langle y,x\rangle = 0$.

Proof. If X is an inner-product space, then the generalized inner product is the inner-product and therefore $\langle x,y\rangle = 0 \implies \langle y,x\rangle = 0$.

The other way, if $\langle x,y\rangle = 0 \implies \langle y,x\rangle = 0$, then by Theorem 3.1.5, Birkhoff-James orthogonality is symmetric. Since the dim $X \geq 3$, X must be an inner-product space (Day [7, Theorem 6.4]).

3.2 Generalized Inner-Product and Characterization of Inner-Product Spaces.

Tapia [46] proved that X must be an inner-product space if the generalized inner-product $\langle x,y\rangle$ is either linear in x or symmetric. Laugwitz [29] gave a geometric proof of the same result. In the following we provide another proof.

Theorem 3.2.1. For a normed linear space X, the following are equivalent :

(i) X is an inner-product space.

(ii) $||x|| = ||y|| \implies \lim_{n \to \infty} (||nx+y|| - ||x+ny||) = 0$

(iii) $\langle x,y\rangle = \langle y,x\rangle$ for all $x,y \in X$.

(iv) $\langle x,y\rangle$ is linear in x for each $y \in X$.

Proof. (i) ==> (ii) is straight.

(ii) ==> (iii). Let $||x|| = ||y||$. Then

$$\begin{aligned}
\langle x,y\rangle &= ||x||\, q_+'(x,y) \\
&= ||x|| \lim_{n\to\infty} \frac{||x+\frac{1}{n}y|| - ||x||}{\frac{1}{n}} \\
&= ||y|| \lim_{n\to\infty} ||nx+y|| - ||ny|| \\
&= ||y|| \lim_{n\to\infty} (||nx+y||-||x+ny||+||x+ny||-||ny||) \\
&= ||y|| \lim_{n\to\infty} (||ny+x|| - ||ny||) \\
&= ||y||\, q_+'(y,x) = \langle y,x\rangle .
\end{aligned}$$

If $||x|| \neq ||y||$, then $||\, ||x||\, y|| = ||\, ||y||\, x||$ and the above argument yields

$$\begin{aligned}
\langle x,y\rangle &= ||x||\, q_+'(x,y) = q_+'(x,||x||y) \\
&= q_+'(||y||x,\ ||x||y) = q_+'(||x||y,\ ||y||x) \\
&= ||y||\, q_+'(y,x) = \langle y,x\rangle .
\end{aligned}$$

(iii) ==> (iv). We first prove that the Birkhoff-James orthogonality is symmetric. Let $x \perp_J y$. Then $-x \perp_J -y$ and $x \perp_J -y$. Therefore $q_+'(-x,-y) \geq 0$ and $q_+'(x,-y) \geq 0$. Thus

$$q_+'(-y,x) = \frac{||x||}{||y||} q_+'(x,-y) \geq 0$$

and

$$q_+'(-y,-x) = \frac{||x||}{||y||} q_+'(-x,-y) \geq 0.$$

Therefore $-y \perp_J x$, and hence $y \perp_J x$.

Next we show that Birkhoff-James orthogonality is right unique ($0 \neq x$, $x \perp_J ax+y$, $x \perp_J bx+y \Longrightarrow a = b$).

If not, then there exist $x,y \in X$, $||x|| = ||y|| = 1$ and number $\epsilon > 0$ such that

$x \perp_J y$ and $x \perp_J ax+y$ for all $0 \leq a \leq \epsilon$ (see James [20, Theorem 2.3]). That means

$$||ax+y+kx|| \geq ||ax+y|| \quad \text{for all } k \in R.$$

From this and by using symmetry of Birkhoff-James orthogonality we have

$$||y|| \geq ||ax+y|| \geq ||y||.$$

Thus

$$(3.2.1) \quad ||x|| = ||y|| = 1 = ||ax+y|| \quad \text{for } 0 \leq a \leq \epsilon.$$

Then by (iii) and (3.2.1)

$$(3.2.2) \quad q'_+(ax+y,x) = q'_+(x,ax+y) = a||x||+q'_+(x,y),$$

and

$$\begin{aligned}(3.2.3) \quad q'_+(ax+y,x) &= \frac{1}{a}\, q'_+(ax+y,ax+y-y) \\ &= \frac{1}{a}\, ||ax+y|| + \frac{1}{a}\, q'_+(ax+y,-y) \\ &= \frac{1}{a}\, ||ax+y|| + \frac{1}{a}\, q'_+(-y,ax+y) \\ &= \frac{1}{a}\, ||ax+y|| - \frac{1}{a}\, ||y|| + q'_+(-y,x) \\ &= q'_+(-y,x) = q'_+(x,-y).\end{aligned}$$

Equations (3.2.2) and (3.2.3) yield

$$a||x|| + q_+'(x,y) = q_+'(x,-y) \quad \text{for } 0 \leq a \leq \epsilon,$$

which is impossible. Thus Birkhoff-James orthogonality is right unique proving that X is smooth (Theorem 1.1.6). Thus for each x,

$$\langle x,y\rangle = ||x||\; q_+'(x,y) \text{ is linear in } y.$$

From this using (iii) we see that

$$a\,\langle x_1,y\rangle + b\,\langle x_2,y\rangle = \langle y,ax_1\rangle + \langle y,bx_2\rangle$$

$$= \langle y,ax_1+bx_2\rangle = \langle ax_1+bx_2,y\rangle.$$

Therefore $\langle x,y\rangle$ is linear in x for each $y \in X$.

(iv) ==> (i).

Let $||x|| = ||y|| = 1$. Then

$$(3.2.4) \quad ||x+y||\; q_+'(x+y,y) = ||x+y||q_+'(x+y,x+y-x)$$

$$= ||x+y||^2+||x+y||\; q_+'(x+y,-x)$$

$$= ||x+y||^2+||x||\; q_+'(x,-x)+||y||q_+'(y,-x)$$

$$= ||x+y||^2-||x||^2+||y||q_+'(y,-x).$$

Also

$$(3.2.5) \quad ||x+y||q_+'(x+y,y) = ||x||\; q_+'(x,y) + ||y||\; q_+'(y,y)$$

$$= ||y||^2 + ||x||\; q_+'\,(x,y).$$

From (3.2.4) and (3.2.5), we have

$$(3.2.6) \qquad ||x+y||^2=||y||^2+||x||^2+||x||q_+'(x,y)-||y||q_+'(y,-x)$$

$$= 2+||x||\ q_+'(x,y) - ||y||\ q_+'(y,-x)\ .$$

Replacing y by -y in (3.2.6), we get

$$(3.2.7) \qquad ||x-y||^2 = 2+||x||q_+'(x,-y) -||y||\ q_+'(-y,-x)$$

$$= 2+||x||\ q_+'(x,-y) + ||y||\ q_+'(y,-x).$$

Adding (3.2.6) and (3.2.7) yields

$$||x+y||^2 + ||x-y||^2 = 4 + (q_+'(x,y)-q_+'(x,-y))$$

$$\geq 4.$$

Thus, if in the space X, (iv) holds, then

$$(S,\sim) \qquad ||x|| = ||y|| = 1 \Longrightarrow ||x+y||^2 + ||x-y||^2 \geq 4,$$

which is a characterization of inner-product spaces due to Schoenberg [38]. That completes the proof of the theorem.

<u>Remark 3.2.2.</u> The implication (ii) of Theorem 3.2.1 is due to James ([20], Theorem 6.3). His proof was different.

<u>Remark 3.2.3.</u> We write $f(x) = \frac{1}{2}||x||^2$. Then $f_+'(x,y) = ||x||\ q_+'(x,y)$. The second right Gâteaux derivative of f at x in the direction of y_1 and y_2 is defined to be

$$f_+''(x;y_1,y_2) = \lim_{t \to 0^+} \frac{||x+ty_1||q_+'(x+ty_1,y_2)-||x||q_+'(x,y_2)}{t}$$

Then

$$f''_{+}(0;y_1,y_2) = \lim_{t \to 0^+} \frac{||ty_1|| \; q'_{+}(ty_1,y_2)}{t} = ||y_1||q'_{+}(y_1,y_2)$$

$$= \langle y_1,y_2 \rangle.$$

Thus Tapia [46] observes that X is an inner-product space if and only if the second right derivative of $\frac{1}{2}||x||^2$ is linear or symmetric (or if f is twice Fréchet differentiable at the origin).

This might be seen in relation to inner-product characterizations by twice Fréchet differentiality of the norm away from the origin given by Bonic and Reis [3], Rao [36], Sundaresan [42] and Leonard and Sundaresan [30].

We give below another characterization of inner-product spaces in terms of differentiability properties of the norm. Let us recall that if X is a smooth normed linear space, then the linear functional $||x|| \; q'_{+}(x,-)$ is denoted by J_x and the map $x \to J_x$ is called the normalized duality map. It is characterized by the properties

$$||J_x|| = ||x|| \quad \text{and}$$

$$J_x(x) = ||x||^2.$$

<u>Theorem 3.2.4.</u> For a normed linear space X, the following are equivalent :

(i) X is an inner-product space.

(ii) X is smooth and $J_{x+y} = J_x + J_y$ whenever $x \perp_J y$.

(iii) X is smooth and $J : X \to X^*$ is linear.

(iv) Whenever $x, y \in X$, $\phi, \psi \in X^*$ are such that

$||\phi|| = ||\psi|| = \phi(x) = \phi(y) = ||x|| = ||y|| = 1$, we have $(\phi+\psi)(x+y) \,\dot{\sim}\, ||x+y||^2$, where $\dot{\sim}$ is one of the relations $\geq$, $=$, and $\leq$.

Proof. (i) ==> (ii) is immediate.

(ii) ==> (iii) follows from the fact that the normalised duality map J is odd i.e. $J_{-x} = -J_x$, and an odd orthogonally additive transformation must be linear (Sundaresan [43, Lemma 2]).

(iii) ==> (iv). Let $x, y \in X$ and $\phi, \psi \in X^*$ such that

$||\phi|| = ||\psi|| = 1 = ||x|| = ||y|| = \phi(x) = \psi(y)$. Since X is smooth, $\phi = J_x, \psi = J_y$. Therefore

$$(\phi+\psi)(x+y) = (J_x+J_y)(x+y) = J_{x+y}(x+y) = ||x+y||^2$$

(iv) ==> (i). Let $||x|| = ||y|| = 1$ and let ϕ and ψ be support functionals at x and y respectively. By (iv) we have

$$||x+y||^2 \,\dot{\sim}\, (\phi+\psi)(x+y) = 2 + \phi(y) + \psi(x)$$

and $$||x-y||^2 \,\dot{\sim}\, (\phi-\psi)(x-y) = 2 - \phi(y) - \psi(x).$$

Thus we have for $x, y \in X$,

$$||x|| = ||y|| = 1 \Longrightarrow ||x+y||^2 + ||x-y||^2 \,\dot{\sim}\, 4.$$

That is the criterion $(S, \dot{\sim})$ for inner-product spaces by Schoenberg. That completes the proof of the theorem.

CHAPTER IV

Continuous Orthogonality Vector Spaces

In a paper Gudder and Strawther [15] took the basic properties of Birkhoff-James orthogonality as axioms to define " an Orthogonality Vector Space" as mentioned in the introductory chapter of the Thesis. In the following we will modify this definition to introduce the concept of 'Continuous Orthogonality Vector Space'. It turns out to be an orthogonality vector space of a special kind. We then obtain a sort of converse to their result [15, Theorem 3.1] characterizing orthogonally increasing functionals on a normed linear space.

A vector space X will be called a Continuous Orthogonality Vector Space (COVS, in short) if there is a relation $x \perp y$ on X satisfying :

(O_1) $x \perp 0$, $0 \perp x$ for all $x \in X$.

(O_2) if $x \perp y$ and $x \neq 0$, $y \neq 0$, then x and y are linearly independent.

(O_3) if $x \perp y$, then $ax \perp by$ for all $a, b \in R$.

(O_4) if P is a two-dimensional subspace of X, then for every $0 \neq x \in P$, there exists a unique $0 \neq y \in P$ (upto scalar multiples) such that $x \perp y$.

(O_5) Whenever $x_n \in P$, $y_n \in P$, $x_n \to x$, $y_n \to y$, and $x_n \perp y_n$, then $x \perp y$ (convergence here is in the conventional topology of P).

It is easily seen that (O_4) is equivalent to

(O_4)' for $x \neq 0$, $y \in X$, there exists a unique α such that

$$x \perp \alpha x + y.$$

Theorem 4.1.1. Let X be a COVS. Define a real valued function ϕ on $X \times X$ as follows :

(i) $\phi(x,y) = 0$ when $x = 0$;

(ii) $\phi(x,y) = \alpha$ when $x \neq 0$ where α is the unique number for which $x \perp \alpha x + y$.

Then

$$\phi(x,y) = 0 \iff x \perp y.$$

Proof. Easy and is omitted.

Theorem 4.1.2. The function ϕ of Theorem 4.1.1 has the following properties :

(i) $\phi(ax,by) = \frac{b}{a}\phi(x,y)$ for all x and $y \in X$ and for all nonzero a, $b \in R$.

(ii) $\phi(x,ax+y) = -a + \phi(x,y)$

$= a\,\phi(x,x) + \phi(x,y)$ for nonzero x and $y \in X$.

Proof.(i) Let $\phi(x,y) = \alpha$, then $x \perp \alpha x + y$ and by (O_3)

$$ax \perp \alpha bx + by$$

which implies that

$$\phi(ax,by) = \alpha \frac{b}{a} = \frac{b}{a} \quad \phi(x,y) \text{ provided } a \neq 0 \text{ and } b \neq 0.$$

(ii) For $x \neq 0$ and $y \neq 0$,

$$\begin{aligned} \phi(x, ax+\cancel{b}y) = \beta &\iff x \perp (\beta+a)\ x + y \\ &\iff \phi(x,y) = \beta + a \\ &\iff \beta = -a + \phi(x,y) \\ &\qquad = a\ \phi(x,x) + \phi(x,y). \end{aligned}$$

<u>Theorem 4.1.3</u>. Let x be a nonzero element of a COVS X and $p,q > 0$. There exists $z \in X$ such that $x \perp z$ and $x + pz \perp x - qz$.

Proof. Let $y \neq 0$ such that $x \perp y$. Set

$$\begin{aligned} g(\mu) &= \phi(x+p\mu y,\ x-q\mu y) \quad \text{for } 0 \leq \mu < \infty \\ &= \phi(x+p\mu y,\ x+p\mu y - (p+q)\mu y) \\ &= -1 + \phi(x+p\mu y,\ -(p+q)\mu y) \\ &= -1 - \frac{(p+q)}{p\mu}\ \mu\ \phi(\frac{x}{p\mu} + y, y) \text{ for } \mu \neq 0 \\ &= -1 - \frac{p+q}{p}\ \phi(\frac{x}{p\mu} + y,\ y) \qquad \text{for } \mu \neq 0. \end{aligned}$$

Thus

$$\lim_{\mu \to \infty} g(\mu) = -1 + \frac{p+q}{p} = \frac{q}{p} > 0.$$

Obviously $g(0) = -1$.

If possible, let $g(\mu) \neq 0$ for all $0 \leq \mu < \infty$. Let $\mu_n \to \mu$ and $g(\mu_n) \to \pm\infty$.

By $(O_4)'$ for each n, there exists $g(\mu_n)$ such that

$$x + p\mu_n y \perp g(\mu_n)\ (x+p\mu_n y) + x-q\mu_n\ y$$

Or $$x + p\mu_n\ y \perp x+p\mu_n y + \frac{x - q\mu_n y}{g(\mu_n)}, \text{ by}(O_3).$$

But then as $n \to \infty$, (O_5) gives

$$x + p\mu y \perp x + p\mu y$$

which is a contradiction. Therefore g is bounded.

Again let $\mu_n \to \mu$ and $g(\mu_{n_i}) \to A$ while $g(\mu_{n_j}) \to B$.

By (O_5), we have

$$x + p\mu y \perp A(x+p\mu y) + (x-q\mu y)$$

and $$x + p\mu y \perp B(x+p\mu y) + (x-q\mu y).$$

Therefore $A = B$ by $(O_4)'$ which implies that $g(\mu_n)$ is a convergent sequence.

Let $g(\mu_n) \to A$. Then we have

$$x + p\mu y \perp A(x+p\mu y) + (x-q\mu y), \text{ whence}$$

$$A = \phi(x+p\mu y,\ x-q\mu y) = g(\mu).$$

Thus g is continuous, $g(0) = -1$ and $\lim_{\mu\to\infty} g(\mu) = \frac{q}{p} > 0$; but $g(\mu) \neq 0$ for all $0 \leq \mu < \infty$, which contradicts Intermediate Value Theorem. Thus

$$g(\mu) = 0 \quad \text{for some } 0 \leq \mu < \infty.$$

But then

$$x + p\mu y \perp x - q\mu y \quad \text{for some } 0 \leq \mu < \infty.$$

Replacement of μy by z, gives the result.

Remark 4.1.4. From Theorem 4.1.3 we see that (O_5) holds in a COVS and therefore any continuous orthogonality vector space is an 'Orthogonality Vector Space'.

Remark 4.1.5. (O_1) to (O_4) may not suffice to yield (O_5) can be seen from the following example :

Example 4.1.6. Let X be R^2 with the orthogonality relation defined as follows :

$$(1,\lambda) \perp (\lambda,1)$$

$$\text{and} \quad (\lambda,1) \perp (1,\lambda) \quad \text{for } -1 < \lambda < 1;$$

$$(1,1) \perp (1,0)$$

$$(-1,1) \perp (1,0)$$

and extend this relation to yield a homogeneous relation.

An inner-product $(\cdot,\cdot)$ on an orthogonality vector space (OVS) is said to be orthogonally equivalent if

$$x \perp y \iff (x,y) = 0$$

A norm $||\cdot||$ on an orthogonality vector space $(X,\perp)$ is said to be orthogonally equivalent if

$$x \perp y \iff x \perp_J y.$$

It has been proved by Gudder and Strawther that if there exists a nontrivial orthogonally additive hemi-continuous even functional on an OVS $(X, \perp)$, then there is an orthogonally equivalent inner product on $(X, \perp)$. In the same paper, they characterize orthogonally increasing functionals on a normed linear space with the orthogonality relation of Birkhoff and James.

Let X be a normed linear space. A function

$$f : X \to R$$

is <u>radially increasing</u> if $\alpha > 1$ implies $f(\alpha x) \geq f(x)$ for all $x \in X$ and f is <u>spherically increasing</u> if

$$||x|| > ||y|| \Longrightarrow f(x) \geq f(y) \text{ for all } x,y \in X.$$

The Theorem of Gudder and Strawther characterizing orthogonally increasing functionals can be stated as follows :

<u>Theorem 4.1.7.</u> (Gudder and Strawther) Let X be a normed linear space with $\dim X \geq 2$ and f be an orthogonally increasing functional on X. Then f is spherically increasing and there exists a countable number of spheres $S_1, S_2, \ldots$ such that f is norm continuous at ω if and only if $\omega \notin \cup S_i$. Furthermore there exists a nondecreasing function

$$g : R^+ \to R$$

such that

$$f(\omega) = g(||\omega||) \text{ for } \omega \notin \cup S_i.$$

In particular if f is an orthogonally increasing continuous functional, then there exists a nondecreasing functional

$$g : R^{+} \to R$$

such that

$$f(\omega) = g(||\omega||).$$

In what follows we show that in a way existence of orthogonally increasing functional on a COVS forces the existence of a norm on X such that the orthogonality is Birkhoff-James orthogonality arising out of that norm. More precisely

Theorem 4.1.8. Let X be a continuous orthogonality vector space (COVS) and

$$f : X \to R$$

such that

(i) $f(\lambda x) = |\lambda| \, f(x)$ for $\lambda \in R$;

(ii) f is orthogonally increasing.

Then f is a norm on X. Further

(iii) if f is Gâteaux differentiable, then the norm f is orthogonally equivalent on $(X, \perp)$.

Firstly we prove two Lemmas :

Lemma 4.1.9. Let X be a COVS and

$$f : X \to R$$

an orthogonally increasing function with $f(0) = 0$. Then f is radially increasing.

Proof. Let $\lambda > 1$. Choose $y \in X$ such that $x \perp y$ and $\frac{1}{\lambda-1} x+y \perp x-y$. Such a $y \in X$ exists by Theorem 4.1.3. Now

$$\begin{aligned} f(\lambda x) &= f\left[(\lambda-1)\left(\frac{1}{\lambda-1} x + y\right) + (\lambda-1)(x-y)\right] \\ &\geq f\left[x + (\lambda-1) y\right] \\ &\geq f(x) \quad \text{for } \lambda > 1. \end{aligned}$$

Thus f is radially increasing.

<u>Lemma 4.1.10.</u> Let X be a COVS and

$$f : X \to R,$$

a nontrivial orthogonally increasing functional with

$$f(\lambda x) = |\lambda|\, f(x) \quad \text{for } \lambda \in R.$$

Then $$f(x) > 0 \quad \text{for } x \neq 0.$$

Proof. Let $x\ (\neq 0) \in X$ such that $f(x) > 0$ and $y(\neq 0) \in X$. Choose $z \neq 0$ in the span of x and y such that

$$x \perp z \text{ and } x+z \perp x-z.$$

Now

$$f(2z) = f(x+z-x+z) \geq f(x+z) \geq f(x) > 0,$$

which implies that

$$f(z) > 0.$$

Let $$y = ax + bz \neq 0. \text{ Then}$$

$$f(y) \geq f(ax) = |a|\, f(x) > 0 \quad \text{if } a \neq 0,$$

and if $a = 0$, $b \neq 0$, then

$$f(y) = |b|\ f(z) > 0.$$

That completes the proof.

Proof of the Theorem. By the two lemmas, we have

(i) $f(x) > 0$ for $x \neq 0$

(ii) $f(x) = 0 \iff x = 0$ and

(iii) $f(\lambda x) = |\lambda|\ f(x)$ for all $\lambda \in R$.

Now it remains to prove the triangle inequality only, so that f may be a norm.

Let $x, y \in X$. If

$$x+y \perp x-y,$$

then

$$f(2x) = f(x+y+x-y) \geq f(x+y)$$

and $$f(2y) = f(y+x+y-x) \geq f(x+y).$$

Hence $f(x+y) \leq f(x) + f(y)$.

If $x + y \not\perp x-y$, let α be such that

$$x + y \perp \alpha(x+y) + (x-y) = (\alpha+1)\ x + (\alpha-1)y.$$

Then

$$f(2y) = f\ [(\alpha+1)(x+y) - (\alpha+1)\ x - (\alpha-1)y]$$

$$\geq f[(\alpha+1)(x+y)] = |\alpha+1|\ f(x+y)$$

and if $a = 0$, $b \neq 0$, then

$$f(y) = |b|\ f(z) > 0.$$

That completes the proof.

Proof of the Theorem. By the two lemmas, we have

(i) $f(x) > 0$ for $x \neq 0$

(ii) $f(x) = 0 \iff x = 0$ and

(iii) $f(\lambda x) = |\lambda|\ f(x)$ for all $\lambda \in R$.

Now it remains to prove the triangle inequality only, so that f may be a norm.

Let $x, y \in X$. If

$$x+y \perp x-y,$$

then

$$f(2x) = f(x+y+x-y) \geq f(x+y)$$

and $f(2y) = f(y+x+y-x) \geq f(x+y)$.

Hence $f(x+y) \leq f(x) + f(y)$.

If $x + y \not\perp x-y$, let α be such that

$$x + y \perp \alpha(x+y) + (x-y) = (\alpha+1)\ x + (\alpha-1)y.$$

Then

$$f(2y) = f\ [(\alpha+1)(x+y) - (\alpha+1)\ x - (\alpha-1)y]$$

$$\geq f[(\alpha+1)(x+y)] = |\alpha+1|\ f(x+y)$$

. and

$$f(2x) = f(-2x) = f\,[(\alpha-1)x + (\alpha-1)y - (\alpha+1)x - (\alpha-1)y]$$

$$\geq f\,[(\alpha-1)(x+y)] = |\alpha-1|\; f(x+y).$$

Hence

$$2f(x+y) \leq (|\alpha+1| + |\alpha-1|)\; f(x+y) \leq f(2x) + f(2y)$$

or $\quad f(x+y) \leq f(x) + f(y).$

Thus f is a norm on X.

By (O_3) $\quad x \perp y \implies x \perp \lambda y$ for all $\lambda \in R$

$$\implies f(x+\lambda y) \geq f(x) \text{ for all } \lambda \in R.$$

Thus the orthogonality is contained in the Birkhoff-James orthogonality of the normed space (X,f).

Suppose that the function f is now Gâteaux differentiable. Let $x \perp_J y$ and $x \not\perp y$. There exists $\alpha \neq 0$ such that

$$x \perp \alpha x + y$$

and then $\qquad x \perp_J \alpha x + y.$

But that means f is not Gâteaux differentiable, contradicting the hypothesis. Hence $x \perp_J y \implies x \perp y$.

That completes the proof.

CHAPTER V

N-Orthogonality and Nonlinear Functionals on Locally Convex Linear Topological Spaces

Let N be a nonlinear mapping of a Hausdorff locally convex linear topological space X into its dual X^*. If x and y are elements of X, x is said to be N-orthogonal to y ($x \perp_N y$, in short) whenever the value of Nx evaluated at y, denoted by (Nx,y) is zero. A real valued function on X is said to be N-orthogonally additive if

$$f(x+y) = f(x) + f(y) \quad \text{whenever } x \perp_N y.$$

In this chapter we will first study this orthogonality and give sufficient conditions on N in order that N-orthogonality becomes nontrivial in the sense that each two-dimensional subspace of X contains a pair of nonzero N-orthogonal elements. We then consider the problem of concretely representing the class of orthogonally additive functionals on X. Sundaresan and Kapoor [44] considered this problem taking N as a linear mapping. Sundaresan [43] discussed the representation of such functionals with Birkhoff-James orthogonality on X. Gudder and Strawther [15] deal with the same problem in the abstract setting of orthogonality vector spaces. At the end of the chapter it is shown that in a locally convex space X if an N-orthogonality satisfies a kind of Pythagoreous property,

then X must be an inner-product space.

5.1 N-Orthogonality

Let $N : X \to X^*$ be a nonlinear mapping. x is called N-orthogonal to $y (x \perp_N y)$ if $(Nx,y) = 0$. Let us recall that the orthogonality is called left (right) homogeneous if $x \perp_N y \Longrightarrow ax \perp_N y$ $(x \perp_N y \Longrightarrow x \perp_N ay)$ for all $a \in R$. It is called symmetric if $x \perp_N y \Longrightarrow y \perp_N x$. The mapping N will be called symmetric if

$$(Nx,y) = (x,Ny) \text{ for all } x,y \in X.$$

It is easily seen that a symmetric mapping is linear. A nonlinear mapping N which gives rise to symmetric orthogonality is given in Example 5.1.2. It is clear that $\perp_N$ is right homogeneous and right additive, but in general it is not left homogeneous or left additive (see Examples 5.1.3 and 5.1.4 below). But if the orthogonality is symmetric then it is both left homogeneous and left additive.

We will assume that $N(0) = 0$ so that 0 is N-orthogonal to every vector in X and every vector in X is orthogonal to zero. We shall further assume that the orthogonality is positive left homogeneous i.e. $tx \perp_N y$ whenever $x \perp_N y$ and $t \geq 0$. This happens to be a fairly strong condition on the mapping N, as seen in the following

Theorem 5.1.1. Let X be a locally convex linear topological space and $N : X \to X^*$. Let R^+ be the set of

nonnegative real numbers. Then N-orthogonality is positive left homogeneous if and only if for some real valued function

f on $R^+ \times X$ which is such that $f(t,0) \equiv 0$ and

$f(t,x) = 0 \iff t = 0$ for $x \neq 0$,

$N(tx) = f(t,x)\, N(x)$ holds.

Further, if $N(x,x) > 0$ for all $x \neq 0$, then

$$f(t,x) \geq 0 \text{ for all } t \in R^+ \text{ and } x \in X.$$

Proof. If $N(tx) = f(t,x)\, Nx$, then $(Nx,y) = 0 \implies (N(tx),y) = 0$ for $t \geq 0$. Therefore the orthogonality is positive left homogeneous. On the other hand if $x = 0$, set $f(t,0) \equiv 0$ and $x \neq 0$ and $t = 0$, set $f(0,x) = 0$. If $x \neq 0$ and $t > 0$, then

$$(Nx,y) = 0 \iff (N(tx),y) = 0.$$

The functionals Nx and $N(tx)$ have the same null space. Hence $N(tx)$ is a nonzero scalar multiple of Nx. Call this multiple $f(t,x)$. Then

$$N(tx) = f(t,x)\, Nx \text{ holds for all } t \in R^+ \text{ and } x \in X.$$

Further let $(Nx,x) > 0$ for all $x \neq 0$. Let $t > 0$, $x \neq 0$. Then $tx \neq 0$ and

$$0 < (N(tx),tx) = t\, f(t,x)\, (Nx,x)$$

which implies

$$f(t,x) > 0.$$

<u>Example 5.1.2.</u> Let H be a Hilbert space. Define

$$N : H \to H^{*}(= H) \text{ as follows :}$$

$$N(x) = ||x||\ x.$$

x is N-orthogonal to y <==> x is orthogonal to y in the Hilbert space and

$$(Nx,y) = ||x||\ (x,y)$$

where (x,y) is the inner-product of x and y.

The N-orthogonality is symmetric but the mapping N is not symmetric.

<u>Example 5.1.3.</u> Let $N : R^2 \to R^2$ given by

$$N(x_1,x_2) = (x_1^3\ ,\ x_2^3).$$

It can easily be seen that N-orthogonality is left homogeneous but not left additive.

<u>Example 5.1.4.</u> Define $N : R^2 \to R^2$ by

$$N(x_1, x_2) = (|x_1|,\ x_2).$$

The N-orthogonality is positive left homogeneous but not left homogeneous.

<u>5.2</u> Existence of Orthogonal Elements

It is clear that if $y_n \to y$ and $x \perp_N y_n$, then $x \perp_N y$. It would be nice if we also have :

$$x_n \to x \text{ and } x_n \perp_N y \implies x \perp_N y.$$

For that we need a continuity condition on N. A function

$$F : X \to X^*$$

will be called hemicontinuous if for x,y and z in X,

$$(N(x+ty),z) \to (Nx,z) \text{ as } t \to 0.$$

We shall obtain a few results on the existence of certain pairs of orthogonal elements in each two-dimensional subspace of X.

Theorem 5.2.1. Let $N : X \to X^*$ be a hemi-continuous mapping and let $(Nx,x) > 0$ for $x \neq 0$. Let the N-orthogonality be positive left homogeneous and let x,y be linearly independent elements of X. Then there exist numbers b and c such that

$$x \perp_N bx + y \text{ and } cx + y \perp_N x.$$

Proof. For the first part take $b = \frac{-(Nx,y)}{(Nx,x)}$. For the second part we have to show that there exists a number c such that

$$(N(cx+y), x) = 0.$$

If $(Ny,x) = 0$, take $c = 0$. If $(Ny,x) < 0$, define

$F(\lambda) = (N(\lambda x+y),x)$, $\lambda \geq 0$, so that

$F(0) = (Ny,x) < 0$, and as $N(tx) = f(t,x)\, Nx$

where f is the function occurring in Theorem 5.1.1 ,

therefore

$$F(t) = f(t,\ x + \tfrac{y}{t})\ (Nx + \tfrac{y}{t}\ ,\ x).$$

Since $(Nx,x) > 0$ for $x \neq 0$, we have

$$f(t, x + \tfrac{y}{t}) > 0.$$

Suppose $F(t) < 0$ for all $t > 0$. By hemi-continuity of N

$$(N(x + \tfrac{y}{t}), x) \to (Nx, x) \leq 0 \text{ as } t \to \infty,$$

which contradicts the hypothesis. Hence

$$F(t) \geq 0 \text{ for some } t > 0.$$

By continuity of F on R, there exists $c > 0$ such that $F(c) = 0$. Then $cx + y \perp_N x$.

On the otherhand if $(Ny,x) > 0$, we get

$$G(t) = (N(y-tx),x) \text{ for } t \geq 0.$$

We then have $G(0) > 0$ and $G(t) = f(t,-x + \tfrac{y}{t})(N(-x + \tfrac{y}{t}),\ x)$.

Suppose $G(t) \geq 0$ for all t. Then we have

$$(N\ (-x + \tfrac{y}{t}),\ x) \geq 0.$$

Taking the limit as $t \to \infty$, we get

$$(N\text{-}x,\ x) \geq 0.$$

But $(N\text{-}x,x) < 0$, because $(N(-x),\ -x) > 0$. Thus $G(t) < 0$ for some $t > 0$. By continuity of G we have $G(d) = 0$ for some d. Set $c = -d$ and then

$$cx + y \perp_N x$$

which completes the proof.

A condition of the type $(Nx,x) \geq 0$ is needed for the existence of N-orthogonal pairs is seen from the following simple example :

<u>Example 5.2.2.</u> Let $N : R^2 \to R^2$ be defined by $N(x_1,x_2) = (x_1^2,x_2^2)$. Take $x = (1,1)$ and $y = (1,2)$. Then

$$(N(cx+y),x) = (c+1)^2 + (c+2)^2 \neq 0 \text{ for all } c \in R.$$

Let $N : X \to X^*$. The mapping N is called a monotone mapping if

$$(Nx-Ny,\ x-y) \geq 0 \quad \text{for all } x \text{ and } y \in X.$$

<u>Theorem 5.2.3.</u> Let $N : X \to X^*$ be a hemi-continuous and monotone mapping and suppose that the N-orthogonality is symmetric. Let x and y be nonzero elements of X such that $x \perp_N y$. Let a be a positive number. Then there exists a number $b > 0$ such that

$$ax - by \perp_N x+bay.$$

Proof. Let t be a positive number. Then $x+tay \neq 0$ and $(N(\frac{y}{a}),\ x + tay) \neq 0$. Therefore by Theorem 5.2.1, there exists $s \in R$ such that

$$(N(x+ty+s\,\tfrac{y}{a}),\ (x+tay)) = 0$$

or

$$(N(ax-dy),x+tay) = 0,$$

where $d = -(at+s)$.

Thus for each $t > 0$ there is a $d \in R$ such that

$$(N(ax-dy),\ x + tay) = 0.$$

We then have

$$d = (N(x+tay),\ ax) \,/\, (N(x+tay),y)$$
$$= (N(ay + \tfrac{x}{t}),ax) \,/ (N(ay + \tfrac{x}{t}),y).$$

Therefore as $t \to 0$, $d \to \infty$ and

as $t \to \infty$, $d \to 0$.

Hence $d(t) = t$ for some $t > 0$. Thus there exists a number b such that

$$(N(ax-by),\ x + bay) = 0.$$

In particular when $a = 1$, we have the result that for some $b \in R$

$$x - by \perp_N x + by.$$

5.3 Representation of Orthogonally Additive Functionals

In this section we consider the problem of representing orthogonally additive functionsl on X. We shall again assume that N is hemi-continuous and $(N(x),x) > 0$ for $x \neq 0$ and $N(0) = 0$.

Theorem 5.3.1. Let $N : X \to X^*$ be a hemi-continuous mapping with the property $(Nx,x) > 0$ for $x \neq 0$, and let the N-orthogonality be homogeneous and nonsymmetric. Let f be a continuous N-orthogonally additive functional on X. Then f is linear.

Proof. Let x and y be two elements of X such that $x \perp_N y$ and $y \not\perp_N x$. Choose $a \in R$ such that $y \perp_N ay+x$. Then for all t and s in R

$$(t+s)\,y + \frac{tx}{a} = sy + (\tfrac{t}{a})\,(ay + x)$$

and

$$f\left(\frac{tx}{a}\right) + f((s+t)y) = f(sy) + f(ty) + f\left(\frac{tx}{a}\right)$$

and therefore

$$f((s+t)y) = f(sy) + f(ty) \text{ for all } t \text{ and } s.$$

The continuity of f implies that f is homogeneous in the direction of y. Choose $a \neq 0$ such that

$$ax + y \perp_N x \text{ (use Theorem 5.2.1).}$$

Since $x \not\perp_N ax+y$, proceeding as above we prove homogeneity of f in the direction of x.

Now let $z_1 = s_1x + t_1y$ and $z_2 = s_2x + t_2y$ be in the span of x and y. Then we have

$$\begin{aligned} f(z_1+z_2) &= f((s_1+s_2)x) + f((t_1+t_2)y) \\ &= f(s_1x) + f(s_2x) + f(t_1y) + f(t_2y) \\ &= f(s_1x + t_1y) + f(s_2x + t_2y) \\ &= f(z_1) + f(z_2) \end{aligned}$$

and $f(tz_1) = f(ts_1x+tt_1y) = f(ts_1x) + f(tt_1y)$

$$= t(f(s_1x) + f(t_1y)) = t\, f(z_1).$$

Now let z be a vector which is not in the span of x and y. We can assume that $x \perp_N z$ and $y \perp_N z$, otherwise we choose numbers s and t such that $x \perp_N sx + ty+z$ and $y \perp_N sx+ty+z$

$$\left[\text{Set } s = \frac{-(Nx,z)}{(Nx,x)} \text{ and } t = \frac{-[(Ny,z) + s(Ny,x)]}{(Ny,y)}\right]$$

and work with $sx + ty + z$ for z in the following.
If $z \not\perp_N x$ or $z \not\perp_N y$, then as before we have f is homogeneous in the direction of z. If $z \perp_N x$ and $z \perp_N y$, in addition to $x \perp_N z$ and $y \perp_N z$, then there is a number b such that $(N(by+z),x) \neq 0$. If not, then

$$(N(y + \tfrac{z}{b}), x) = 0 \text{ for all } b > 0.$$ Taking the limit as $b \to \infty$ we obtain

$$(Ny,x) = 0,$$

which is false. Set

$$c = - (N(by+z),z) \,/\, (N(by+z), x).$$ Then

$by + z \perp_N cx + z$ and therefore

$$\begin{aligned} f((t+s)z) + f(sby+tcx) &= f(sby + sz+tcx+tz) \\ &= f(sby + sz) + f(tcx+tz) \\ &= f(sby) + f(sz) + f(tcx)+f(tz). \end{aligned}$$

Hence once more

$$f((s+t)z) = f(sz) + f(tz).$$

Combining the above results we have

$$f(ax+by+cz) = a\, f(x) + b\, f(y) + c\, f(z) \text{ for all}$$
$$a, b \text{ and } c \text{ in } R \text{ and all } z \in X.$$

Thus f is linear on the span of x,y and z. If there is an element w in X which is not in the span of x,y and z, then proceeding as above, we can prove that f is linear in the span

of x, y, z and w . Therefore f is linear on the span of z and w for arbitrary linearly independent vectors z and w . Hence f is linear.

Theorem 5.3.2. Let $N : X \to X^*$ be a hemicontinuous monotone mapping. Let N-orthogonality be symmetric. Then an odd continuous functional is N-orthogonally additive if and only if it is linear.

Proof. It is easy to verify the 'if' part. Let f be an N-orthogonally additive functional. We first prove that f is homogeneous. Let $y \neq 0$ be such that

$$(Nx, y) = 0 .$$

and choose b by Theorem 5.2.3 such that

$$x - by \perp_N x + by.$$

Then

$$f(2x) = f(x-by+x+by) = f(x-by) + f(x+by)$$

$$= f(x) - f(by) + f(by) + f(x) = 2f(x).$$

Assume that for $m \leq n$,

$$f(mx) = mf(x) \text{ for all } x.$$

Choose b by Theorem 5.2.3 again so that

$$nx + by \perp_N x - n\,by.$$

Then

$$\begin{aligned} f((n+1)x) - f((n-1)by) &= f((nx+by) + (x-nby)) \\ &= f(nx+by) + f(x-nby) \\ &= f(nx)+f(by)+f(x)-f(nby). \end{aligned}$$

Hence

$$f((n+1)x) - (n-1)\ f(by) = n\ f(x) + f(by) + f(x) - nf(by),$$

that is

$$f((n+1)x) = (n+1)\ f(x).$$

Therefore by induction, for all n and all x, we have

$$f(nx) = nf(x);$$

and then we can prove that

$$f(x/n) = \frac{1}{n} f(x) \text{ for all natural numbers } n.$$

The function f is odd and continuous. Hence

$$f(tx) = t\ f(x) \text{ for all } t.$$

Now we show that

$$f(x+y) = f(x) + f(y) \text{ for all } x \text{ and } y \text{ in } X.$$

When $(Nx,y) = 0$, we have nothing to prove. Let $(Nx,y) \neq 0$. Suppose $x \perp_N ax + y$. Then

$$\begin{aligned} f(x+y) &= f(x+(ax+y) - ax) = f((1-a)x + (ax+y)) \\ &= f((1-a)x) + f(ax+y) = (1-a)\ f(x) + f(ax+y) \\ &= f(x) + f(-ax) + f(ax+y) \\ &= f(x) + f(y). \end{aligned}$$

Thus f is linear.

<u>Remark 5.3.4.</u> It can easily be seen that under the hypothesis of the Theorem 5.3.3. X is an orthogonality vector

space in the sense of Gudder and Strawther [15] and therefore the Theorem 5.3.3 also follows from (Lemma 2.1, [15]).

In earlier papers (Sundaresan [43], Sundaresan and Kapoor [44], Gudder and Strawther [15]) it has been proved that

(i) In an inner-product space an even continuous orthogonally additive functional f must be of the form

$$f(x) = c||x||^2 \text{ for some } c \in R.$$

(ii) In locally convex space X an even continuous T-orthogonally additive functional f must be of the form

$$f(x) = c(Tx,x)$$

where the T-orthogonality arises from a linear transformation T of E into E^*, and is symmetric.

Clearly such a result can be carried over to the present situation only if the form

$F(x) = (Nx,x)$ is itself N-orthogonally additive

i.e. if

(*) $(N(x+y), x+y) = (Nx,x) + (Ny,y)$ whenever $(Nx,y) = 0$.

We shall say that N-orthogonality satisfies <u>Pythagorus property</u> whenever N satisfies (*).

The following question then arises naturally :

Does there exist a nonlinear monotone mapping $N : X \to X^*$ for which the N-orthogonality is symmetric and satisfies the Pythagorus property?

To answer this question we bring in the concept of semi-inner-product spaces introduced by Lumer [32], and continuous semi-inner-product spaces by Giles [14]. These have been defined in Chapter I of this Thesis.

<u>Theorem 5.3.5.</u> If X is a locally convex space and X^* is the dual of X and $N : X \to X^*$ is a mapping satisfying the following properties :

(i) $(Nx,x) > 0$ and $(Nx,x) = 0 \iff x = 0$,

(ii) N is hemi-continuous,

(iii) N-orthogonality satisfies the Pythagorus property,

(iv) $N(tx) = t\,N(x)$ for $x \in X$ and $t \in R$,

then X is an inner-product space in the sense that the N-orthogonality is an inner product orthogonality, and N is a linear transformation.

Proof. Define $[x,y] = (Nx,y)$. It is easily verified that $[x,y]$ is a semi-inner-product on X; that it is a continuous semi-inner product having the homogeneity property follows from conditions (ii) and (iv) respectively. In fact

$$|\,[x,y]\,|^2 \leq [x,x]\,[y,y]$$

can be seen as follows :

If $(Nx,y) = 0$, we have nothing to prove. If

$(Nx,y) \neq 0$, choose b such that $x \perp_N bx + y$.

The Pythagorus property yields

$$\begin{aligned}(Ny,y) &= (N(bx+y-bx),\ bx+y-bx)\\ &= (N(bx+y),\ bx+y) + (N(-bx,\ -bx))\\ &= b^2\ (Nx,x) + (N(bx+y),\ bx+y)\\ &\geq b^2\ (Nx,x) = (Nx,y)^2/(Nx,x)\end{aligned}$$

because

$b = -(Nx,y)/(Nx,x)$. Therefore

$$(Ny,y)\ (Nx,x) \geq (Nx,y)^2, \text{ i.e.}$$

$$|\,[x,y]\,|^2 \leq [x,x]\ [y,y].$$

By Theorem 1.1.16, the N-orthogonality is the Birkhoff-James orthogonality in the norm

$$[x,x]^{1/2} = (Nx,x)^{1/2}$$

on X.

By (iii) Birkhoff-James orthogonality implies Pythagorean orthogonality. But then by Theorem 2.1.4, the norm $(Nx,x)^{1/2}$ must be an inner-product norm and N-orthogonality is an inner-product orthogonality. It also follows that N is linear. That completes the proof.

Finally we have

<u>Theorem 5.3.6.</u> If $N : X \to X^*$ satisfies the condition of Theorem 5.3.5 and if F is N-orthogonally additive functional on X, then there exists a linear functional $f \in X^*$ and a

number c such that

$$F(x) = f(x) + c(Nx,x) \text{ for all } x \in X.$$

Proof. By Theorem 5.3.5. N is linear and the rest follows from Sundaresan and Kapoor([44], Theorem 5).

The following example shows that $(Nx,x) > 0$ for $x \neq 0$ is essential in the above results.

Example 5.3.7. Define $N : R^2 \to R^2$ by

$$N(x_1,x_2) = ((x_1+x_2)^3,(x_1+x_2)^3).$$

If $x = (x_1,x_2)$ and $y = (y_1,y_2)$, then

$$N(x,x) = (x_1+x_2)^4$$

and

$$(Nx,y) = (y_1+y_2)\ (x_1+x_2)^3,\ N(y,x) = (x_1+x_2)(y_1+y_2)^3$$

Clearly N-orthogonality is symmetric for

$$(Nx,y) = 0 \iff y_1+y_2 = 0 \text{ or } (x_1+x_2) = 0 \iff (Ny,x) = 0$$

Let $x \perp_N y$, then we have

$$\begin{aligned}(N(x+y),x+y) &= (x_1+x_2+y_1+y_2)^4\\ &= (x_1+x_2)^4 = (Nx,x) + (Ny,y) \text{ if } y_1+y_2 = 0\\ &= (y_1+y_2)^4 = (Nx,x) + (Ny,y) \text{ if } x_1+x_2 = 0.\end{aligned}$$

Therefore N-orthogonality satisfies the Pythagorus property. But the functional $F : R^2 \to R$ defined by

$$F(x) = (x_1 + x_2)^2$$

is even N-orthogonally additive yet is not of the form $c(Nx,x)$ and the functional

$$g : R^2 \to R$$

$$g(x) = (x_1+x_2)^3$$

is odd N-orthogonally additive functional, yet it is not linear.

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